Lecture Notes on

Quantum Entanglement and von Neumann Algebras

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These are lecture notes for a course in October 2026 at Perimeter Institute, offered as module two of the course QIC891 at the University of Waterloo. For further information, including dates and locations of the lectures, see

https://lauritz-van-luijk.github.io/teaching

The course is based on a series of works with Alexander Stottmeister, Henrik Wilming, and Reinhard F. Werner [1–7]. We started working actively on the topic in 2023 while I was a PhD student in Hanover, and I ended up writing my PhD thesis about this subject.

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1 Introduction

Entanglement lies at the heart of quantum information theory, underpinning a range of phenomena from cryptography and quantum computation to topological order [8–12]. The structure and manipulation of entanglement are well understood in finite systems. Although many settings of foundational and physical interest involve infinitely many degrees of freedom, a systematic operational study of entanglement in these settings has only recently been undertaken.

In systems with infinitely many degrees of freedom, states can be infinitely entangled across a pair of subsystems. Examples range from the vacuum in relativistic quantum field theories [13–15] to ground states in infinite spin chains [16]. This leads us to questions like:

Are there different kinds of infinite entanglement?
If so, can we distinguish them operationally?

To address these questions systematically, we need to talk about how we model quantum systems and subsystems with infinitely many degrees of freedom. We also need to specify the operations that an agent with access to a subsystem is assumed able to implement. This is not just a question about the mathematical model. More importantly, it is a question of which physical and operational setting we consider.

For concreteness, we commit to a specific class of physical models, *quantum spin systems in the thermodynamic limit* [17]. Moreover, we make the physically motivated choice that implementable operations may only act non-trivially on finitely many sites, see Fig. 1. For instance, on a one-dimensional spin chain, the operation that applies a fixed unitary u_0 at every site of a finite interval in one of the half-chains is implementable, but if u_0 acts at every site of the half-chain, the operation is not considered implementable by an agent. However, we will consider *approximate state transitions* instead of exact ones: We regard a state transition

$$\psi \to \phi$$

as implementable by an agent with access to a subsystem A if they can use finite operations to make the error vanish *uniformly* in expectation values [6]. Mathematically, this amounts to convergence *in norm* (in finite dimensions, this norm is the trace norm). We summarize this operational setting as:

finite operations and approximate state transitions.

As a consequence, we may restrict our attention to fixed *sectors* of the system. Sectors are certain norm-closed subsets of the state space of a quantum many-body system, which are closed under finite operations. For any pure state ψ, the set of states $\mathcal{S}_\psi$ that are reachable with operations on finite regions

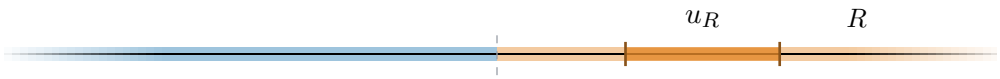
$$\mathcal{S}_\psi = \{\phi : \psi \xrightarrow{\Gamma} \phi\} \tag{1}$$

is a sector. The GNS representation of ψ identifies $\mathcal{S}_\psi$ with the set of density operators on the GNS Hilbert space $\mathcal{H}_\psi$. We restrict ourselves to sectors that contain pure states, ruling out thermal sectors. Such sectors are necessarily of the form (1) for each pure state ψ that they contain.

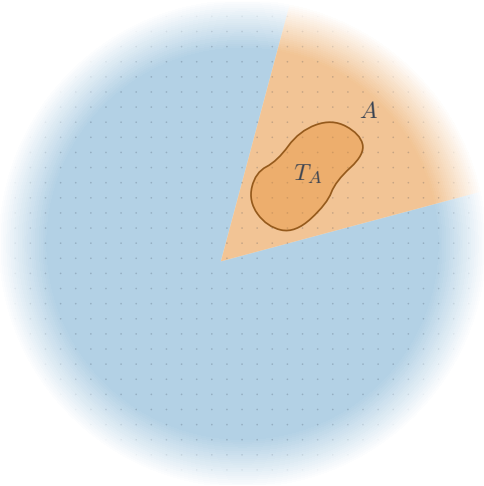
Working in a pure state sector, we may use the corresponding GNS Hilbert space $\mathcal{H}$. We can take suitable topological completions of the algebras of finitely localized operators in a given region, without changing the possible approximate state transitions. This gives von Neumann algebras associated to subsystems: For a region A,

$$\mathcal{M}_A = \overline{\{\text{operators acting on a finite subregion of } A\}}^{sot}, \tag{2}$$

where the closure is in the strong operator topology. If we keep the fixed sector and only ask information-theoretic questions involving approximate state transitions with finite operations, the answers will be determined completely by the Hilbert space $\mathcal{H}$ and the von Neumann algebras $\mathcal{M}_{A_j}$ corresponding to the subsystems A_j that we consider.



(a) 1D spin chain: A unitary u_R acts on a finite interval of the right half-chain R.



(b) 2D spin system: A quantum channel T_A acts on a finite subregion of a cone A.

Figure 1: Operations on finite regions of infinite spin systems.

We can therefore study information-theoretic questions entirely in terms of von Neumann algebras, but this does not yet explain why we should do so. The main reasons are that it is mathematically natural and that it is economical: If two bipartitions of two unrelated quantum spin systems lead to unitarily equivalent triples $(\mathcal{M}_A, \mathcal{M}_B, \mathcal{H})$ in the sectors under consideration, then both setups will have the same entanglement properties. Thus, by studying entanglement theory for abstract triples $(\mathcal{M}_A, \mathcal{M}_B, \mathcal{H})$ of two commuting von Neumann algebras on a Hilbert space $\mathcal{H}$, we can work on bipartite entanglement theory for infinite quantum systems in a model-independent way.

The past years have shown that this point of view is quite fruitful [1, 5, 6]. Operational entanglement properties are directly encoded in the algebraic properties and relative positions of the von Neumann algebras describing the subsystems. Conversely, deep algebraic properties can be determined from information-theoretic properties of quantum systems [1, 18, 19].

Although we restrict our physical applications to infinite quantum spin systems, the results that we obtain are general connections between information-theoretic and algebraic properties. They can be applied whenever modeling subsystems by von Neumann algebras and subsystem-implementable operations by quantum channels with Kraus operators in those algebras makes sense. For instance, this includes continuous systems in quantum statistical mechanics.

The idea that quantum information-theoretic properties of systems with infinitely many degrees of freedom should be understood through algebraic properties of von Neumann algebras first appeared in works of Summers and Werner in the 1980s [14, 20–22], where they studied Bell's inequalities in quantum field theory.

Material for learning von Neumann algebras. I am aware that the theory of von Neumann algebras can be hard for an outsider to grasp. We will cover the basics of what we need, emphasizing intuition over details, but it will be impossible to give a good account of most of the theory. What you should not do if you want to learn more about von Neumann algebras is open Takesaki's books and just start reading from the beginning. These books are often the standard reference in papers, but they are more of an encyclopedia than anything else. I personally like to use and recommend F. Hiai's lecture notes [23]. I also recommend V. Jones' lecture notes [24]. J. Sorce has a nice exposition aimed at the high-energy physics community [25]. My PhD thesis [26] also has an chapter introducing von Neumann algebras and several parts of the theory.

Acknowledgments. I am very grateful for this opportunity to teach my work, especially to Debbie Leung who made this possible against several administrative odds.

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AI usage. The lecture notes are entirely written by myself. The figures, except for Fig. 2, were created in TikZ using AI. I used AI to proofread and to correct typos and small mistakes. I also used it to obtain first drafts of the last two sections based on my PhD thesis and our papers, but then I completely rewrote both sections.

2 Some basics about von Neumann algebras

The theory of von Neumann algebras was born in a series of papers [27–32] by Murray and von Neumann in the 1930s. One of von Neumann's motivations for developing this theory is worth emphasizing here: He was unsatisfied with the formulation of quantum mechanics in terms of Hilbert spaces, which he developed earlier, because it was not singled out by a physical principle among more general descriptions, in terms of what we call von Neumann algebras today [33]. Indeed, he was right! Systems with infinitely many degrees of freedom like those arising in field theory or quantum statistical mechanics cannot be modeled by Hilbert spaces alone, but require von Neumann algebras [17, 34]. This is precisely why we care about von Neumann algebras here.

A metaphor often used to describe von Neumann algebras is that they represent "noncommutative measure spaces" [35]. More specifically, the idea is that a von Neumann algebra is the L^∞-space of a "noncommutative measure space". This metaphor goes far beyond the fact that abelian von Neumann algebras are isomorphic to actual L^∞-spaces over measure spaces. Indeed, almost all concepts in measure theory transfer to the noncommutative setting in some way. Examples include monotone convergence theorems, L^p-spaces, and Radon-Nikodym theorems. I personally believe this metaphor to be useful.[1] It is also instructive to compare the use of von Neumann algebras in describing quantum systems to classical statistical mechanics, where many questions are about probability theory (=measure theory).

2.A Infinite-dimensional Hilbert spaces and operator topologies. We consider a separable infinite-dimensional Hilbert space $\mathcal{H}$. Separability means there exists a countable orthonormal basis, and since changing the basis is a unitary, all separable infinite-dimensional Hilbert spaces are

[1] There exists a companion metaphor for C*-algebras: They describe "noncommutative topological spaces". Again, there is much more to this than abelian C*-algebras being isomorphic to the continuous functions on a compact Hausdorff space. We refer interested readers to Connes' book [35].

isomorphic. All Hilbert spaces we consider will be separable (including finite-dimensional Hilbert spaces).

The algebra $B(\mathcal{H})$ of bounded operators[2] on $\mathcal{H}$ is our first example of a von Neumann algebra. We will see more examples later on, but need to understand $B(\mathcal{H})$ a little better first. In particular, we need to discuss some topologies on $B(\mathcal{H})$.

The *weak* operator topology is the topology of matrix elements, i.e., convergence $x_n \to x$ in this topology is equivalent to

$$\langle \xi, x_n \eta \rangle \to \langle \xi, x\eta \rangle \qquad \forall \xi, \eta \in \mathcal{H}. \tag{3}$$

An equivalent way to define convergence in the weak operator topology is via

$$\mathrm{tr}(\rho x_n) \to \mathrm{tr}(\rho x) \tag{4}$$

for all *finite-rank* density operators ρ (a density operator is a positive operator, $\rho \geq 0$, of unit trace, $\mathrm{tr}(\rho) = 1$). If one instead allows for arbitrary density operators, the defined topology is called the *σ-weak* (also called the *ultraweak*) operator topology. Convergence in the *strong operator topology* is defined by

$$x_n \xi \to x\xi \qquad \forall \xi \in \mathcal{H}. \tag{5}$$

Here, convergence of vectors in $\mathcal{H}$ means norm convergence, i.e., $\|x_n\xi - x\xi\| \to 0$.

A linear functional $\phi : B(\mathcal{H}) \to \mathbb{C}$ is called *normal* if it is ultraweakly continuous, and the set of normal linear functionals is just the trace-class $\mathcal{T}(\mathcal{H})$ in disguise: A linear functional ϕ is normal if and only if there is a trace-class operator ρ such that

$$\phi(x) = \mathrm{tr}(\rho x). \tag{6}$$

In particular, each density operator corresponds to a normal (algebraic) state, i.e., a normal positive linear functional $\phi : B(\mathcal{H}) \to \mathbb{C}$ with $\phi(1) = 1$.

On subsets of $B(\mathcal{H})$ that are bounded in operator norm, the weak and ultraweak operator topologies coincide. Indeed, every density operator can be approximated in trace norm by finite-rank density operators, and boundedness makes the approximation uniform. The strong operator topology is strictly finer, even on the unit ball.

2.B Von Neumann algebras. Let us begin with the definition. A *-algebra $\mathcal{M} \subseteq B(\mathcal{H})$ is a complex subspace that is closed under the adjoint and the product. It is unital if $1 \in \mathcal{M}$.

Definition 2.1. *A unital *-algebra $\mathcal{M} \subseteq B(\mathcal{H})$ is a von Neumann algebra if it is closed in the strong operator topology:*

$$\mathcal{M} = \overline{\mathcal{M}}^{sot}. \tag{7}$$

If $X \subseteq B(\mathcal{H})$ is a self-adjoint subspace, then its commutant

$$X' = \{y \in B(\mathcal{H}) \ : \ xy = yx, \ x \in X\} \tag{8}$$

is a von Neumann algebra. Note that one always has $X \subseteq X''$.

Many of the basic properties of von Neumann algebras are encoded in von Neumann's *bicommutant theorem*, asserting that a unital *-algebra $\mathcal{M} \subseteq B(\mathcal{H})$ is strongly closed, i.e., is a von Neumann algebra, if and only if it is its own bicommutant:

$$\mathcal{M} = \mathcal{M}''. \tag{9}$$

[2]"bounded" means that the operator norm $\|x\| = \sup\|x\xi\|$, where the sup is over all unit vectors ξ, is finite.

The bicommutant theorem tells us that von Neumann algebras always come in pairs $(\mathcal{M}, \mathcal{M}')$.

Another consequence is that a von Neumann algebra contains the *spectral projections* and the *functional calculi* of each of its normal elements. Indeed, this follows from (9) because the spectral projections and functional calculi of a normal operator a commute with all operators that commute with a.

In (7), we defined von Neumann algebras using the strong operator topology. Perhaps surprisingly, one obtains an equivalent definition by taking the weak or the ultraweak operator topology (although these topologies are not equivalent). This follows from duality results for Banach spaces that we will not discuss here, see, e.g., [36]. To summarize, we have the following equivalent characterizations of what a von Neumann algebra is:

Theorem 2.2. *Let $\mathcal{M} \subseteq B(\mathcal{H})$ be a unital self-adjoint *-algebra. The following are equivalent:*

(a) $\mathcal{M}$ is strongly closed, i.e., a von Neumann algebra;

(b) $\mathcal{M}$ is weakly closed;

(c) $\mathcal{M}$ is ultraweakly closed;

(d) $\mathcal{M}$ is its own bicommutant, i.e., $\mathcal{M} = \mathcal{M}''$.

2.C States and the predual. Coming from quantum theory, we are used to identifying the states of a quantum system described by a Hilbert space $\mathcal{H}$ with the set of density operators on $\mathcal{H}$. For a general von Neumann algebra $\mathcal{M} \subseteq B(\mathcal{H})$, one cannot define states as "density operators in $\mathcal{M}$". One could consider equivalence classes of density operators on $\mathcal{H}$, where two density operators are equivalent if they induce the same expectation values on $\mathcal{M}$, but there is a more elegant approach using the algebraic point of view:

Definition 2.3. *A normal state on a von Neumann algebra $\mathcal{M}$ is an ultraweakly continuous positive linear functional $\phi : \mathcal{M} \to \mathbb{C}$ with $\phi(1) = 1$.*

The role of the partial trace is then taken over by mere restriction: If ϕ is a normal state on $B(\mathcal{H})$, its $\mathcal{M}$-marginal is the normal state $\phi|_{\mathcal{M}}$ on $\mathcal{M}$. This elegantly packages the equivalence mentioned above: Two density operators ρ, ρ' on $\mathcal{H}$ induce the same expectation values on $\mathcal{M}$ precisely when the corresponding algebraic states ϕ, ϕ' have equal $\mathcal{M}$-marginals $\phi|_{\mathcal{M}} = \phi'|_{\mathcal{M}}$. Moreover, every normal state ϕ on a von Neumann algebra $\mathcal{M} \subseteq B(\mathcal{H})$ is induced by a density operator ρ on $\mathcal{H}$ [37, Thm. 7.1.9, Rem. 7.1.10].

The span of the normal states on a von Neumann algebra is the set of all normal (=ultraweakly continuous) linear functionals $\mathcal{M} \to \mathbb{C}$. This space is known as the *predual* of $\mathcal{M}$ and is denoted $\mathcal{M}_*$. It is a Banach space in its natural norm

$$\|\phi\| = \sup_{\substack{a \in \mathcal{M} \\ \|a\| \leq 1}} |\phi(a)|, \qquad \phi \in \mathcal{M}_*, \tag{10}$$

and the dual space of $\mathcal{M}_*$ is isomorphic to $\mathcal{M}$ via the dual pairing $(a, \phi) \mapsto \phi(a)$.

2.D Inner quantum channels. For any family of "Kraus" operators $k_1, k_2, \ldots \in \mathcal{M}$ with $\sum_i k_i^* k_i = 1$, we obtain a completely positive normalization-preserving map $T : \mathcal{M}_* \to \mathcal{M}_*$

$$T(\phi) = \sum_i k_i \phi k_i^*, \tag{11}$$

where $k\phi k^*$ is defined as the functional $a \mapsto \phi(k^*ak)$. In the Heisenberg picture, the dual map T^* acts as

$$T^*(a) = \sum_i k_i^* a k_i. \tag{12}$$

The maps T and T^* are called *inner* quantum channels on $\mathcal{M}$, and all of this is true for infinite families of Kraus operators.[3,4]

2.E The set of projections in a von Neumann algebra. Much of the early structure theory for von Neumann algebras rests on analyzing properties of projections. In fact, the type classification that we will discuss later is formulated this way (except for the subtypes of type III).

Projections in $B(\mathcal{H})$, i.e., elements with $p = p^* = p^2$, are in one-to-one correspondence with closed subspaces $V \subseteq \mathcal{H}$ via $p\mathcal{H} = V$. The projection onto the complementary subspace $V^\perp$ is $1-p$. We use the convention to denote the projection onto the closed subspace generated by $S \subseteq \mathcal{H}$ by $[S]$.

We begin by characterizing which closed subspaces correspond to projections in a given von Neumann algebra $\mathcal{M}$.

Lemma 2.4. *Let $\mathcal{M}$ be a von Neumann algebra on $\mathcal{H}$ and let $p = [V]$ be the projection onto a closed subspace $V \subseteq \mathcal{H}$. Then $p \in \mathcal{M}$ if and only if $\mathcal{M}'V \subseteq V$.*

I strongly recommend stopping and trying to come up with the proof yourself. You can find a written proof below.

Often the algebraic point of view is cleaner. For instance, the statement that a closed subspace V is $\mathcal{M}'$-invariant if and only if its orthogonal complement $V^\perp$ is feels more complicated than the equivalent statement $p \in \mathcal{M} \iff 1-p \in \mathcal{M}$ for a projection p.

Using this lemma, we can derive some nice properties of the set of projections of a von Neumann algebra. Projections are ordered by the usual positivity order $p \leq q \iff 0 \leq q-p$. For subspaces, this order corresponds to simple inclusion $V \subseteq W$. For any family of closed subspaces V_i, $i \in I$, there exist a smallest closed subspace containing each V_i, namely $\vee_i V_i := \overline{\operatorname{span} \cup_i V_i}$, the closed linear subspace generated by the V_i, and a largest closed subspace contained in each V_i, namely their intersection $\cap_i V_i$. We denote the corresponding operations for projections p_i by $\vee_i p_i$ and $\wedge_i p_i$. The two operations are related via De Morgan's rule

$$(\vee_i V_i)^\perp = \cap_i V_i^\perp \qquad \iff \qquad 1 - \vee_i p_i = \wedge_i (1-p_i). \tag{13}$$

As a consequence, we have:

Lemma 2.5. *A von Neumann algebra is closed under infima and suprema of projections.*

Again, I recommend trying to prove this yourself. It is a relatively simple exercise that teaches you to work algebraically.

Proof of Lemma 2.4. Suppose $p \in \mathcal{M}$. If $a' \in \mathcal{M}'$, then $a'p = pa'$ gives $a'V = a'p\mathcal{H} = pa'\mathcal{H} \subseteq p\mathcal{H} = V$. Conversely, $\mathcal{M}'V \subseteq V$ gives $pa'p = a'p$ for all $a' \in \mathcal{M}'$. Taking adjoints also shows $pa'p = pa'$ for all $a' \in \mathcal{M}'$, so that $pa' = a'p$. Hence, $p \in \mathcal{M}''$. The bicommutant theorem gives $p \in \mathcal{M}$. □

[3] There are more general normal unital completely positive maps $\mathcal{M} \to \mathcal{M}$ that are not inner, i.e., do not admit a Kraus representation in $\mathcal{M}$.

[4] The condition $\sum_i k_i^* k_i = 1$ implies that (11) and (12) converge.

Proof of Lemma 2.5. Given projections p_i, $i \in I$, set $V_i = p_i\mathcal{H}$. If each V_i is $\mathcal{M}'$-invariant, then so is their intersection $\cap_i V_i$. By Lemma 2.4 this gives $\wedge_i p_i \in \mathcal{M}$, i.e., the greatest lower bound of p_i, $i \in I$, is a projection in $\mathcal{M}$. By De Morgan's rule, the corresponding statement for $\vee$ follows from the one for $\wedge$. □

As an application, we can define the support projection of a normal state on a von Neumann algebra. Let ω be a normal state on $\mathcal{M}$. Then the support projection is

$$\operatorname{supp}(\omega) := \inf \{p \in \mathcal{M} \text{ projection} : \omega(p) = 1\}. \tag{14}$$

Lemma 2.6. *Let ρ be a density operator on $\mathcal{H}$ and let $\omega(a) = \operatorname{tr}(\rho a)$, $a \in \mathcal{M}$, be the induced normal state on $\mathcal{M}$. Then*

$$\operatorname{supp}\omega = [\mathcal{M}'\rho\mathcal{H}]. \tag{15}$$

In particular, if $\rho = |\Omega\rangle\langle\Omega|$, then $\operatorname{supp}\omega = [\mathcal{M}'\Omega]$.

Proof. Let $s = [\mathcal{M}'\rho\mathcal{H}]$. By Lemma 2.4, $s \in \mathcal{M}$. Moreover, $s\rho = \rho$ so that $1 = \operatorname{tr} s\rho = \omega(s)$. Conversely, let $p \in \mathcal{M}$ be a projection with $\omega(p) = 1$. Let $\rho = \sum_j \lambda_j |\Psi_j\rangle\langle\Psi_j|$ be the spectral decomposition of ρ. Then

$$0 = \omega(1-p) = \sum_j \lambda_j \|(1-p)\Psi_j\|^2, \tag{16}$$

so that $(1-p)\Psi_j = 0$ or, equivalently, $\Psi_j \in p\mathcal{H}$, for all j with $\lambda_j > 0$. Thus, $p\rho = \rho$. Since p commutes with $\mathcal{M}'$ this implies that it acts as the identity on $\mathcal{M}'\rho\mathcal{H}$. Hence, $s \leq p$. Consequently, the infimum in (14) is attained at $p = s$. □

2.F Factors. The intersection of a von Neumann algebra and its commutant is the center of $\mathcal{M}$:

$$Z(\mathcal{M}) := \mathcal{M} \cap \mathcal{M}'. \tag{17}$$

A von Neumann algebra is called a *factor* if its center is trivial. Note that $\mathcal{M}$ is a factor if and only if $\mathcal{M}'$ is. The name alludes to the fact that $\mathcal{M}$ is a factor if and only if $B(\mathcal{H})$ "factors" into $\mathcal{M}$ and $\mathcal{M}'$ in the sense that

$$B(\mathcal{H}) = \mathcal{M} \vee \mathcal{M}', \tag{18}$$

where $\vee$ denotes the generated von Neumann algebra. If the Hilbert space decomposes as a tensor product $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, then

$$\mathcal{M} = B(\mathcal{H}_A) \otimes 1, \qquad \mathcal{M}' = 1 \otimes B(\mathcal{H}_B) \tag{19}$$

is a factor. Initially, it was guessed that every factor was of this form [38, p. 63]. Factors of the form (19) are called type I factors. They form the easiest examples of factors.

Factors are the building blocks of general von Neumann algebras. If M is a von Neumann algebra on a separable Hilbert space $\mathcal{H}$, there is an essentially unique so-called direct integral decomposition into factors (this uses the assumption that the Hilbert space is separable). Direct integrals are continuous generalizations of direct sums, e.g., $L^\infty[0,1]$ can be expressed as a direct integral of $\mathbb{C}$ over $[0,1]$.

2.G Classification into types and subtypes. The classification of factors into types I_n ($n \in \mathbb{N}\cup\{\infty\}$), II_1, II_∞, and III is as old as the theory of von Neumann algebras itself. It was established by Murray and von Neumann in the pioneering papers [27, 29]. The classification of type III factors is due to Connes [39], but the way we present it here is due to [40].

A projection q is orthogonal to a projection p if $p+q$ is a projection or, equivalently, if $q \leq 1-p$. The classification of factors rests on the so-called *Murray-von Neumann comparison theory* of projections: Projections p and q are said to be equivalent, written $p \sim q$, if there exists an operator $v \in \mathcal{M}$ such that $p = v^*v$ and $q = vv^*$, and one says that p dominates q, written $q \preceq p$, if $q \leq p' \sim p$ for some p'. An operator v such that vv^*, v^*v are projections is called a *partial isometry*.

A projection p is called ...

- *finite* if $p \sim q \leq p \implies p = q$;
- *minimal* if $\quad q \leq p \implies q = p$ or $q = 0$.

A von Neumann algebra is called finite if the unit $1 \in \mathcal{M}$ is a finite projection.

The type classification of factors goes as follows [36] (see Fig. 2 for a visualization):

- Type I: The factor contains a nonzero minimal projection. Type I factors are further sub-classified into:
  - Type I_n, $n \in \mathbb{N}\cup\{\infty\}$: The maximal number of mutually orthogonal nonzero projections is n. Up to isomorphism, every type I_n factor is isomorphic to $\mathcal{B}(\mathcal{H})$ for an n-dimensional Hilbert space $\mathcal{H}$.
- Type II: The factor contains nonzero finite projections but no nonzero minimal projections. Type II factors are sub-classified into:
  - Type II_1 : The unit $1 \in \mathcal{M}$ is a finite projection.
  - Type II_∞: The unit $1 \in \mathcal{M}$ is not a finite projection.
- Type III: The factor contains no nonzero finite projections. Equivalently, $p \sim q$ for all pairs of nonzero projections $p, q \in \mathcal{M}$. Type III factors are further sub-classified into:
  - Type III_λ, $0 \leq \lambda \leq 1$: The value of λ is defined in terms of the *diameter of the state space* modulo unitary equivalence [40, 41]

$$\operatorname{diam} \mathcal{S}_*(\mathcal{M})/_\sim \;:=\; \sup_{\psi,\phi\in\mathcal{S}_*(\mathcal{M})} \inf_{u\in U(\mathcal{M})} \|\psi - u\phi u^*\| \tag{20}$$

    where $U(\mathcal{M})$ is the unitary group of $\mathcal{M}$, as

$$\lambda = \left(\frac{2 - \operatorname{diam} \mathcal{S}_*(\mathcal{M})/_\sim}{2 + \operatorname{diam} \mathcal{S}_*(\mathcal{M})/_\sim}\right)^2 . \tag{21}$$

Every factor has exactly one of the specified types and exactly one of the specified subtypes.

The definition of type III_λ factors in terms of the diameter of the state space is taken from [40], where it is shown that $\operatorname{diam} \mathcal{S}_*(\mathcal{M})/_\sim = 2\frac{1-\lambda^{1/2}}{1+\lambda^{1/2}}$. While this is very different from the definitions found in textbooks, which we will discuss later on, it has the benefit of being explicitly stated without introducing any modular theory. Moreover, a nice side-effect is that the diameter-based definition is operational: It simply asks how different states can be modulo unitary equivalence.

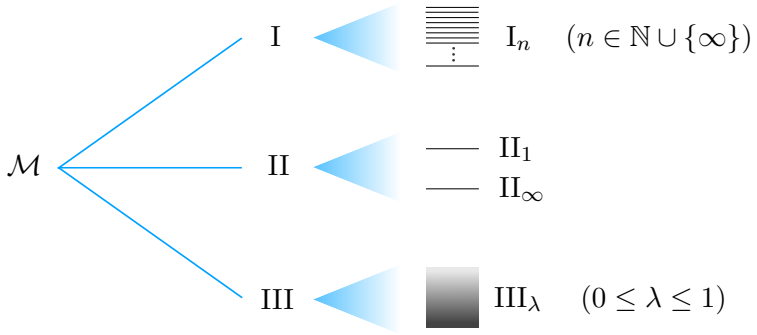


Figure 2: The classification of factors into types I, II, III, and their respective subtypes.

The extreme case is type III_1, where every normal state ψ is approximately unitarily equivalent to any other normal state ϕ.

The types of a factor and its commutant are tightly connected:

type of $\mathcal{M}$	type of $\mathcal{M}'$
I	I
II	II
III_λ, $0 \leq \lambda \leq 1$	III_λ

(22)

If $\mathcal{M}$ is a type I or type II factor, we cannot obtain information about the subtype of the commutant from the subtype of $\mathcal{M}$.

2.H Approximate finite-dimensionality. A von Neumann algebra $\mathcal{M}$ is called approximately finite-dimensional (AFD) if it is generated by an increasing family of finite-dimensional subalgebras. All von Neumann algebras describing subsystems of quantum spin systems are inherently AFD. Moreover, it seems that all observable algebras used to model quantum systems are AFD.[5]

Factors that are AFD are almost completely classified by the type classification [44]: Two AFD factors are *-isomorphic if and only if they have the same type and subtype, except when they are type III_0 factors. The latter are rather exotic objects that we will not encounter.

2.I W*-algebras (skipped in the lecture). Interestingly, the predual gives rise to an abstract way of defining von Neumann algebras. While this won't be central to our purposes, it is an important fact that should be mentioned here (it can also be used as a basis of a nice physical discussion, when the von Neumann algebra is taken to model a quantum system). A C*-algebra is a (unital) *-algebra that comes with a norm and some axioms, allowing one to prove that a normed unital *-algebra is a C*-algebra if and only if it is isometrically *-isomorphic to a norm-closed unital *-algebra on a Hilbert space $\mathcal{H}$. In contrast, von Neumann algebras are strongly closed unital *-algebras. Therefore, every von Neumann algebra is a C*-algebra. The question arising is which properties of an abstract C*-algebra guarantee that it is *-isomorphic to a von Neumann algebra. The beautiful answer, discovered by Sakai [45], is that a C*-algebra A is *-isomorphic

[5] This is noteworthy for people interested in Tsirelson's problem because violations of Tsirelson's conjecture cannot arise if the A- and B-POVMs live in commuting AFD von Neumann algebras [42, 43]. If it is indeed true that non-AFD algebras do not arise in the description of physical systems, then no bipartition of a quantum system with infinitely many degrees of freedom can give rise to a violation of Tsirelson's conjecture, despite working in the "commuting operator framework".

to a von Neumann algebra $\mathcal{M}$ on a Hilbert space if and only if it has a predual, i.e., if there is a Banach space F, so that A is isometrically isomorphic to the dual F^*. In this case, the *-isomorphism automatically comes from an isometric isomorphism $F \to \mathcal{M}_*$ between the preduals. In the representation-free approach, von Neumann algebras are often called W*-algebras.

This also highlights the conceptual difference when comparing C*-algebraic and von Neumann algebraic probabilistic models: In the C*-algebraic case, observables are primal objects and states are dual objects, and this relation is reversed in von Neumann algebras, where states are primal and observables are dual.

2.J Approximation theorems. The following collects two useful approximation techniques. In the lecture, they are postponed until when we need to apply them.

We show that if a contraction does not change the norm of a vector by much, then it can be approximated by a unitary in the von Neumann algebra.

Lemma 2.7. *Let Ψ be a unit vector, and let $a \in \mathcal{M}$ be a contraction in a von Neumann algebra $\mathcal{M}$. Let $\varepsilon > 0$. If $\|a\Psi\|^2 \geq 1 - \varepsilon$, then there exists a unitary $u \in \mathcal{M}$ with*

$$\|a\Psi - u\Psi\|^2 \leq 2\varepsilon. \tag{23}$$

The argument is based on the Russo-Dye theorem [46], asserting that the set of contractions in a von Neumann algebra is the closed convex hull of the unitary group. The proof is inspired by [41, Lem. 2.4].

Proof of Lemma 2.7. Consider the norm-continuous $\mathbb{R}$-linear map $f = \mathrm{Re}\langle a\Psi, (\,\cdot\,)\Psi\rangle : \mathcal{M} \to \mathbb{R}$ and note that $f(a) = \|a\Psi\|^2$. By the Russo-Dye theorem, the set of contractions C of $\mathcal{M}$ is the closed convex hull of the unitary group $U(\mathcal{M})$. Therefore,

$$\sup_{b \in C} f(b) = \sup_{u \in U(\mathcal{M})} f(u). \tag{24}$$

Take a unitary $u \in \mathcal{M}$ with

$$f(u) \geq \sup_{b \in C} f(b) - \frac{\varepsilon}{2} \geq f(a) - \frac{\varepsilon}{2}. \tag{25}$$

We have

$$\begin{aligned} \|a\Psi - u\Psi\|^2 &= \|a\Psi\|^2 + \|u\Psi\|^2 - 2\mathrm{Re}\langle a\Psi, u\Psi\rangle \\ &= \|a\Psi\|^2 + 1 - 2f(u) \\ &\leq \|a\Psi\|^2 + 1 - 2f(a) + \varepsilon \\ &\leq 1 - \|a\Psi\|^2 + \varepsilon \leq 2\varepsilon. \end{aligned} \tag{26}$$

□

Next, we state Kaplansky's density theorem, which we will use to show that operations acting on finite subregions of quantum spin systems can approximate every operation of the von Neumann algebras associated with a region. In general, this theorem is quite useful. An often recalled quote puts it nicely:

> *The density theorem is Kaplansky's great gift to mankind. It can be used every day, and twice on Sundays.*
>
> — Gert K. Pedersen [47]

Theorem 2.8 (Kaplansky, [36])**.** *Let $\mathcal{D}$ be a unital *-algebra on $\mathcal{H}$ and let $\mathcal{M} = \mathcal{D}''$ be the generated von Neumann algebra. Then the contractions in $\mathcal{D}$ are strongly dense in the contractions of $\mathcal{M}$.*

3 Uniqueness of purifications and Haag duality

A fundamental property in finite-dimensional quantum information theory is the uniqueness of purifications up to unitaries: If $\Psi, \Phi \in \mathcal{H}_A \otimes \mathcal{H}_B$ purify the same density operator ρ_A, there exists a unitary u_B on $\mathcal{H}_B$ such that

$$\Psi = u_B \Phi. \tag{27}$$

Conversely, the existence of such a unitary clearly implies that the two states have the same A-marginal. In the presence of an environment E, purifications of states on A are unique up to unitaries in B if and only if the environment E is trivial, i.e., $\dim \mathcal{H}_E = 1$. Thus, in finite dimensions, uniqueness of purifications is equivalent to the full system being composed of the two subsystems A and B.

Surprisingly, in settings with infinitely many degrees of freedom, uniqueness of purifications is no longer a fundamental property. It can fail, even in settings where the full system is composed of the two subsystems being studied and even when we allow for arbitrarily small errors. The system being composed of two subsystems can be made operationally precise through tomographic completeness: If two states ρ, σ are indistinguishable through correlation experiments performed by A and B, then they have to be equal.

We will prove that uniqueness of purifications is equivalent to Haag duality, asserting that the von Neumann algebras of A and B are commutants of each other. This result was discovered by Alexander Stottmeister, Henrik Wilming, and me in [7]. Haag duality is an important algebraic property, often appearing as a sufficient condition for constructions or proofs [1, 3, 34, 48–50]. The reason that Haag duality is such a convenient property is that it makes the mathematical machinery connecting a von Neumann algebra and its commutant directly available to operational statements; see Sec. 2.

Before we come to the theorem, we consider an example of a bipartitioned quantum spin system, in which we can explicitly understand why uniqueness of purifications fails.

3.A Example: 2-cone regions in topologically ordered systems. The following discussion is based on [7], which is inspired by [51].

We consider a pure state sector $\mathcal{H}$ of a quantum spin system in the infinite 2D plane. We pick a reference state $\Omega \in \mathcal{H}$ and assume the following topological order properties:

- *Transportability.* To every (thickened) line ℓ, one can associate a unitary "line operator" $u_\ell \in \mathcal{M}_\ell$, so that, up to phase, $u_\ell \Omega$ only depends on the homotopy class of ℓ relative to the endpoints. We think of this as creating a charge/anti-charge pair, which are moved along the path ℓ to its endpoints.

- *Detectability.* If a (thickened) closed loop γ encloses exactly one of the endpoints of ℓ, there is a projection $p_\gamma \in \mathcal{M}_\gamma$ with $\langle u_\ell \Omega, p_\gamma u_\ell \Omega \rangle = 1$ and $\langle \Omega, p_\gamma \Omega \rangle = 0$.

We consider the following partition of the plane: Pick a pair of disjoint cones A_1 and A_2 and set $A = A_1 \cup A_2$ and $B = A^c$. The exact geometry of cones is irrelevant in the following. What is important is that A is composed of two disjoint regions that stretch out to infinity.

We create a charge/anti-charge pair and move one into A_1 and the other one into A_2 by picking a line operator u_ℓ with ℓ connecting interior points of A_1 and A_2. Let us denote the resulting excited state by

$$\Psi = u_\ell \Omega. \tag{28}$$

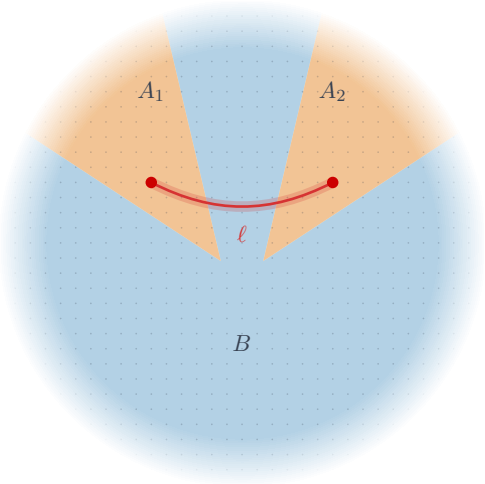


Figure 3: Failure of uniqueness of purifications in 2-cone geometries for topologically ordered systems. The ground state Ω is compared to the state Ψ with anyon/anti-anyon excitations in the cones A_1 and A_2, created by a line operator u_ℓ connecting interior points of the respective cones. Both Ω and Ψ have the same B-marginal, but remain orthogonal under the action of unitaries in $A = A_1 \cup A_2$.

Transportability of the topological charges implies that Ψ and Ω have the same B-marginal. Indeed, let $b \in \mathcal{A}_B^{loc}$ be an operator acting in a finite region $B_0 \Subset B$. If the region does not intersect ℓ, u_ℓ and b commute, so that

$$\langle \Psi, b\Psi \rangle = \langle \Omega, b u_\ell^* u_\ell \Omega \rangle = \langle \Omega, b\Omega \rangle. \tag{29}$$

If the region does intersect ℓ, we can use the transportability and the 2-cone geometry to pick a line operator $u_{\ell'}$ so that ℓ' has the same endpoints and does not intersect B_0. Since $u_\ell \Omega = u_{\ell'} \Omega$ up to phase, we can apply (29) to $u_{\ell'}$. Thus, $\langle \Psi, b\Psi \rangle = \langle \Omega, b\Omega \rangle$ for all $b \in \mathcal{A}_B^{loc}$ and, consequently, for all b in $\mathcal{M}_B$.

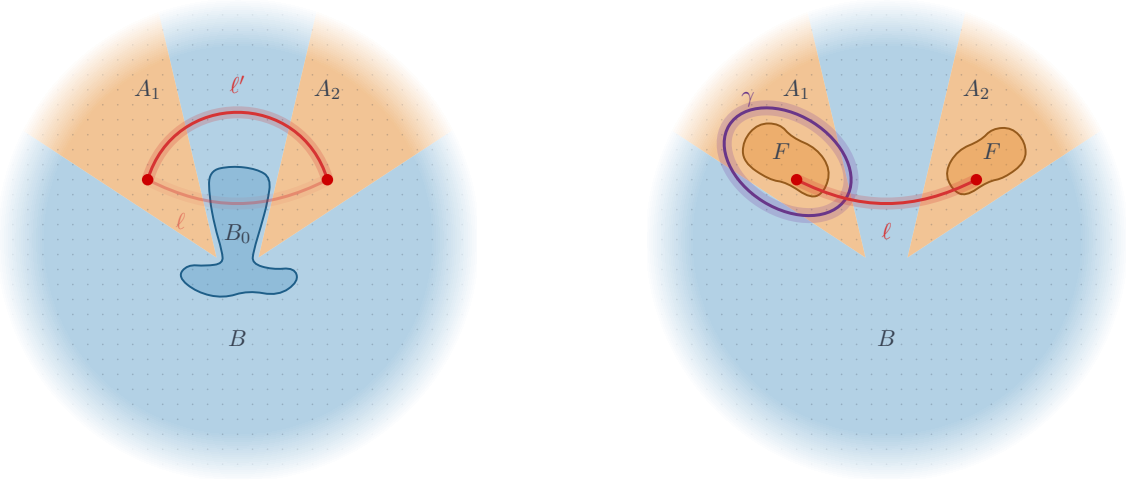


Figure 4: The two geometric steps in the proof. Left: the line ℓ is deformed to ℓ' with the same endpoints, avoiding the finite region $B_0 \subset B$. Right: the loop γ surrounds $F \cap A_1$ and the left endpoint, while remaining disjoint from F and A_2.

The above gives us a pure state Ψ that has the same marginal state on B as the reference state Ω. To disprove the uniqueness of purifications in this example, we show that it is impossible to approximate Ψ by acting on Ω with finite-support unitaries in A. In fact, detectability implies that $u_A\Omega \perp \Psi$ for all unitaries $u_A \in \mathcal{M}_A$. Indeed, for any $a \in \mathcal{A}_A^{loc}$ acting in a finite region $F \Subset A$, we can choose a thickened loop γ encircling both the endpoint of ℓ in A_1 and $F \cap A_1$, but not touching the cone A_2. By the detectability assumption, there is a projection $p_\gamma \in \mathcal{M}_\gamma$ such that $p_\gamma\Psi = \Psi$ and $p_\gamma\Omega = 0$. Since $p_\gamma \in \mathcal{M}_\gamma$ and since γ is disjoint from $F \Subset A$, we know that p_γ and a commute. Therefore, $p_\gamma\Omega = 0$ implies

$$\langle \Omega, a\Psi\rangle = \langle \Omega, ap_\gamma\Psi\rangle = \langle \Omega, p_\gamma a\Psi\rangle = 0 \tag{30}$$

for arbitrary $a \in \mathcal{A}_A^{loc}$. Since $\mathcal{A}_A^{loc}$ is weakly dense in $\mathcal{M}_A$, (30) holds for every $a \in \mathcal{M}_A$. In particular, this shows that Ω is orthogonal to $u_A\Psi$ for any unitary $u_A \in \mathcal{A}_A^{loc}$. We stress that this includes interactions between the cones, i.e., in general, $u_A \neq u_{A_1} \otimes u_{A_2}$.

We summarize our observation: If we bipartition a topologically ordered system into a union of two disjoint cones and its complement, then the reference state Ω and the excited state Ψ, obtained by creating a charge in one cone and its anti-charge in the other cone, have the same marginal state outside of the two cones, but remain orthogonal when acting with local unitaries in the two cones.

3.B Uniqueness of purifications is equivalent to Haag duality. We will prove the following theorem:

Theorem 3.1 ([7]). *Let $\mathcal{M}_A, \mathcal{M}_B$ be commuting von Neumann algebras on a Hilbert space $\mathcal{H}$. The following are equivalent:*

(a) Haag duality: $\mathcal{M}_A = \mathcal{M}_B'$;

(b) if $\Psi, \Phi \in \mathcal{H}$ are unit vectors with $\psi_A = \phi_A$, then $\Psi = v_B\Phi$ for a partial isometry $v_B \in \mathcal{M}_B$;

(c) if $\Psi, \Phi \in \mathcal{H}$ are unit vectors with $\psi_A = \phi_A$, then, for all $\varepsilon > 0$, there exists a unitary $u_B \in \mathcal{M}_B$ such that $\|\Psi - u_B\Phi\| < \varepsilon$.

For the proof we need to establish several facts of general importance.

We start with a construction of a partial isometry from a pair of unit vectors with the same marginal on a von Neumann algebra. The lemma can be generalized to settings where the two vectors live in different representations of the algebra $\mathcal{M}$.

Lemma 3.2. *If $\Psi, \Phi \in \mathcal{H}$ are unit vectors with the same $\mathcal{M}$-marginal $\psi_\mathcal{M} = \phi_\mathcal{M}$, then*

$$v_0 : \mathcal{M}\Psi \to \mathcal{M}\Phi, \quad a\Psi \mapsto a\Phi, \tag{31}$$

extends to a partial isometry $v \in \mathcal{M}'$ with $v^*v = [\mathcal{M}\Psi]$ and $vv^* = [\mathcal{M}\Phi]$.

Proof. Equality of the marginals implies that v_0 is isometric and well-defined:

$$\|v_0a\Psi\| = \|a\Phi\| = \langle \Phi, a^*a\Phi\rangle^{1/2} = \langle \Psi, a^*a\Psi\rangle^{1/2} = \|a\Psi\|.$$

Hence, v_0 extends continuously to a unitary $\overline{\mathcal{M}\Psi} \to \overline{\mathcal{M}\Phi}$, which we again denote by v_0. We have $av_0b\Psi = ab\Phi = v_0ab\Psi$ for $a, b \in \mathcal{M}$. Hence,

$$av_0 = v_0a, \qquad a \in \mathcal{M}. \tag{32}$$

Let p be the projection onto $\overline{\mathcal{M}\Psi}$. Then $p \in \mathcal{M}'$ and we set $v = v_0p$. Then (32) and $p \in \mathcal{M}'$ give

$$av = av_0p = v_0ap = v_0pa = va. \tag{33}$$

Hence, $v \in \mathcal{M}'$. □

To go from exact transitions with partial isometries to approximate transitions with unitaries, we use Lemma 2.7.

Proof of Theorem 3.1. (a) ⇒ (b) is shown in Lemma 3.2. (b) ⇒ (c) is shown in Lemma 2.7.

(c) ⇒ (a): *Step 1.* Let $\Psi \in \mathcal{H}$ be a unit vector. For any unitary $u_{A'} \in U(\mathcal{M}'_A)$, the vector $u_{A'}\Psi$ has the same A-marginal as Ψ. Using our assumption, we obtain $u_{A'}\Psi \in \overline{U(\mathcal{M}_B)\Psi}$. Since $u_{A'}$ was arbitrary, we have

$$U(\mathcal{M}'_A)\Psi \subseteq \overline{U(\mathcal{M}_B)\Psi}.$$

This and the fact that a von Neumann algebra is the linear hull of its unitary group, $\mathcal{M} = \operatorname{span} U(\mathcal{M})$, give $\mathcal{M}'_A\Psi \subseteq \overline{\mathcal{M}_B\Psi}$ and, hence,

$$\overline{\mathcal{M}'_A\Psi} \subseteq \overline{\mathcal{M}_B\Psi}.$$

But $\mathcal{M}_B \subseteq \mathcal{M}'_A$ shows the converse inclusion, so that we actually have $\overline{\mathcal{M}_B\Psi} = \overline{\mathcal{M}'_A\Psi}$. Since Ψ was arbitrary, this shows that every $\mathcal{M}_B$-cyclic projection (these are projections in $\mathcal{M}'_B$, see Lemma 2.4) is in $\mathcal{M}_A$.

Step 2. Take an arbitrary projection p in $\mathcal{M}'_B$. Let $\{\Psi_n\}$ be an orthonormal basis of $p\mathcal{H}$. Then $p\mathcal{H}$ is the closed subspace generated by the subspaces $\mathbb{C}\Psi_n$. Since $p\mathcal{H}$ is $\mathcal{M}_B$-invariant, it is also the closed subspace generated by the $\mathcal{M}_B$-invariant subspaces $\overline{\mathcal{M}_B\Psi_n}$. Let $p_n = [\mathcal{M}_B\Psi_n]$. Then $p = \vee_n p_n$. The first step implies $p_n \in \mathcal{M}_A$. Since von Neumann algebras are closed under suprema and infima of projections, we also have $p \in \mathcal{M}_A$. Therefore, all projections in $\mathcal{M}'_B$ are elements of $\mathcal{M}_A$. Applying this to the spectral projections of arbitrary hermitian elements gives $\mathcal{M}_A = \mathcal{M}'_B$. □

To connect this theorem with the observation of failure of the uniqueness of purifications in suitable bipartitions of topologically ordered systems, we have to explain what the von Neumann algebras describing subsystems of quantum spin systems are and how they model operations of finite support. This was already sketched in the introduction; the next section will give an accurate account.

4 Approximate state transitions in quantum spin systems

In this section, roughly following [6], we show that the operational setting of *finite operations and approximate state transitions* in quantum spin systems admits a description purely in terms of von Neumann algebras, as discussed in the introduction.

We consider an infinite quantum spin system of d-dimensional quantum systems localized at the sites of a lattice $\Gamma \subset \mathbb{R}^k$ in an arbitrary spatial dimension.

We expand the discussion in the introduction, in particular, by giving proofs for the claims we made. We will see how the physically motivated choice of considering approximate state transitions with finitely localized operations implies that quantum information-theoretic questions can be answered in a von Neumann algebraic framework. Essentially, this follows from the following two observations.

1. finitely localized operations cannot change the sector of a state;
2. closing the observable algebra of a subsystem in the strong topology does not change which state transitions are approximately implementable.

We begin with a sector-independent approach to describing the spin system based on the local algebra $\mathcal{A}^{loc}$, containing all operators that act only on finitely many sites. We will then show that approximate state transitions can be formulated entirely in terms of the von Neumann algebraic description of subsystems in fixed sectors.

4.A The local algebra. For each finite subset $A \Subset \Gamma$, we set $\mathcal{H}_A = \bigotimes_{x\in A} \mathbb{C}^d$ and $\mathcal{A}_A := B(\mathcal{H}_A)$. If $A \subseteq B$, tensoring with the identity on $B \setminus A$ gives the natural inclusion $\mathcal{A}_A \hookrightarrow \mathcal{A}_B$. We will simply regard $\mathcal{A}_A$ as a subalgebra of $\mathcal{A}_B$ whenever $A \subseteq B$. This allows us to define $\mathcal{A}^{loc}$ as the union[6]

$$\mathcal{A}^{loc} = \bigcup_{A\Subset\Gamma} \mathcal{A}_A, \tag{34}$$

where $X \Subset Y$ means $X \subseteq Y$ and $\#X < \infty$. For any region $A \subseteq \Gamma$ (not necessarily finite), we can form $\mathcal{A}_A^{loc}$ in the analogous way and we have $A \subseteq B \implies \mathcal{A}_A^{loc} \subseteq \mathcal{A}_B^{loc}$. The *-algebra $\mathcal{A}^{loc}$ carries a natural norm.[7] The completion is a C*-algebra, called the *quasi-local algebra*, $\mathcal{A}$ [52, 53]. It contains elements that act nontrivially on more than finitely many sites, if they can be approximated by finite operators in norm.

The quantum states of the full system can be identified with the algebraic state space $\mathcal{S}(\mathcal{A}^{loc})$. By density of $\mathcal{A}^{loc}$ in $\mathcal{A}$, this is the same as the state space $\mathcal{S}(\mathcal{A})$ of the quasi-local algebra $\mathcal{A}$. To distinguish states, we use the norm distance

$$\|\psi - \phi\| = \sup_{\substack{a\in\mathcal{A}^{loc}\\ \|a\|\leq 1}} |\psi(a) - \phi(a)|. \tag{35}$$

Equivalently, one can optimize over contractions in the quasi-local algebra $\mathcal{A}$.

A state ω on $\mathcal{A}^{loc}$ is pure, i.e., an extremal point of the state space, precisely when $0 \leq \phi \leq \omega$ implies $\phi \propto \omega$. This holds for states on general (pre-)C*-algebras by the same proof as in finite dimensions. I encourage you to prove it yourself.

4.B State transitions. A quantum channel localized in a finite region $A \Subset \Gamma$ is a completely positive map on the state space determined by a collection $K \subset \mathcal{A}_A$ of Kraus operators, i.e., $\sum_{k\in K} k^*k = 1$, via

$$T : \mathcal{S}(\mathcal{A}^{loc}) \to \mathcal{S}(\mathcal{A}^{loc}), \quad \psi \mapsto \sum_{k\in K} k\psi k^*, \tag{36}$$

where $k\psi k^*$ is the positive linear functional defined by $a \mapsto \psi(k^*ak)$. In the Heisenberg picture, the same channel is given by the map $T^* : a \mapsto \sum_{k\in K} k^*ak$ on $\mathcal{A}^{loc}$. For a region $A \subseteq \Gamma$, we write

$$\psi \xrightarrow{A} \phi \tag{37}$$

if, for any $\varepsilon > 0$, there exists a channel T_ε, acting on a finite region in A, such that $\|T_\varepsilon(\psi) - \phi\| < \varepsilon$.

4.C Sectors and their GNS Hilbert spaces. Superselection sectors (sectors for short) are subsets of the state space $\mathcal{S}(\mathcal{A}^{loc})$. Roughly, the idea is that superpositions only make sense for states within a fixed superselection sector, while superposing states from different sectors results in statistical mixtures. We will not define abstract superselection sectors here. Instead, we restrict our

[6] More precisely, the inductive limit, not the union.

[7] If $a \in \mathcal{A}^{loc}$, then $\|a\| := \|a\|_{\mathcal{A}_A}$ is independent of the choice of $A \Subset \Gamma$ with $a \in \mathcal{A}_A = B(\mathcal{H}_A)$. The norm $\|\ \|_{\mathcal{A}_A}$ is just the operator norm on $\mathcal{H}_A$. The resulting norm on $\mathcal{A}^{loc}$ has the C*-property $\|a^*a\| = \|a\|^2$.

attention to sectors that contain pure states. A sector $\mathcal{S}$ containing a pure state ω can be recovered from that pure state as the set

$$\mathcal{S} = \{\phi \in \mathcal{S}(\mathcal{A}^{loc}) : \omega \xrightarrow{\Gamma} \phi\} \tag{38}$$

of states that can be reached with operations on finite regions. For these lectures, we take this as a definition: A (pure state) sector is a subset of the state space of the form (38) for some pure state ω. We will see that this does not depend on the choice of reference pure state.

Next, we describe how to associate a Hilbert space to a sector using the GNS construction. In general, the GNS construction takes an algebraic state ω on a C*-algebra $\mathcal{A}$ (or pre-C*-algebra such as $\mathcal{A}^{loc}$) and associates to it a triple $(\mathcal{H}, \pi, \Omega)$ of a Hilbert space $\mathcal{H}$, a representation $\pi : \mathcal{A} \to B(\mathcal{H})$, and a unit vector $\Omega \in \mathcal{H}$ that purifies ω in the sense that $\omega(a) = \langle \Omega, \pi(a)\Omega\rangle$. We discuss the construction in our specific case. We consider the quotient vector space $\mathcal{A}^{loc}/N$, where $N = \{a \in \mathcal{A}^{loc} : \omega(a^*a) = 0\}$. On $\mathcal{A}^{loc}/N$, we define an inner product via

$$\langle [a], [b]\rangle := \omega(a^*b), \qquad a, b \in \mathcal{A}^{loc}. \tag{39}$$

The completion with respect to the induced norm $\|[a]\| = \omega(a^*a)^{1/2}$ is a Hilbert space

$$\mathcal{H}_\omega := \overline{\mathcal{A}^{loc}/N}. \tag{40}$$

The local algebra $\mathcal{A}^{loc}$ acts on $\mathcal{H}_\omega$ via $\pi(a)[b] = [ab]$. This representation is faithful and isometric.[8] Thus, it extends continuously to a representation of the quasi-local algebra $\mathcal{A}$. Setting $\Omega := [1]$, we have a pure state in $\mathcal{H}_\omega$ that implements ω,

$$\omega(a) = \langle \Omega, \pi(a)\Omega\rangle, \qquad a \in \mathcal{A}^{loc}. \tag{41}$$

Moreover, we have $\pi(a)\Omega = [a]$. By construction, Ω is a cyclic vector for the representation, that is

$$\mathcal{H}_\omega = \overline{\pi(\mathcal{A}^{loc})\Omega}. \tag{42}$$

So far, we have not used purity. Using it, we can show that the GNS representation is irreducible, meaning that the commutant of $\pi(\mathcal{A}^{loc})$ is the scalars. Indeed, if $a' \in \pi(\mathcal{A}^{loc})'$ is such that $0 \le a' \le 1$, then $\phi := \langle \Omega, \pi(\,\cdot\,)a'\Omega\rangle$ is a positive linear functional such that $0 \le \phi \le \omega$. By purity this implies $\phi = t\omega$, $0 \le t \le 1$. Therefore,

$$\langle a'\Omega, \pi(a)\Omega\rangle = \phi(a) = t\omega(a) = t\langle \Omega, \pi(a)\Omega\rangle, \qquad a \in \mathcal{A}^{loc}.$$

Using cyclicity this gives $a'\Omega = t\Omega$. Using cyclicity again, we obtain $a' = t1$ from

$$\langle \pi(a)\Omega, a'\pi(b)\Omega\rangle = \langle \pi(a)\Omega, \pi(b)a'\Omega\rangle = t\langle \pi(a)\Omega, \pi(b)\Omega\rangle, \qquad a, b \in \mathcal{A}^{loc}.$$

By the bicommutant theorem (9), irreducibility can equivalently be expressed as

$$\overline{\pi(\mathcal{A}^{loc})}^{sot} = \pi(\mathcal{A}^{loc})'' = B(\mathcal{H}). \tag{43}$$

Next, we show how the states in the sector $\mathcal{S}$ can be identified with density operators on $\mathcal{H}$. Given a state $\phi \in \mathcal{S}$, we want to show that there exists a unique density operator ρ_ϕ on $\mathcal{H}$ such that

$$\mathrm{tr}(\rho_\phi \pi(a)) = \phi(a), \qquad a \in \mathcal{A}^{loc}. \tag{44}$$

[8] Indeed, an element $a \in \mathcal{A}^{loc}$ only acts on a finite region A and $\mathcal{A}_A \cong B((\mathbb{C}^d)^{\otimes |A|})$, so $\pi|_{\mathcal{A}_A}$ is a representation of $B((\mathbb{C}^d)^{\otimes |A|})$ and hence isometric. Therefore $\|\pi(a)\| = \|\pi|_{\mathcal{A}_A}(a)\| = \|a\|_{\mathcal{A}_A} = \|a\|$. This implies faithfulness.

Uniqueness follows from strong density of $\pi(\mathcal{A}^{loc})$ in $B(\mathcal{H})$; see (43). We show existence. Let $\mathcal{S}_0$ be the set of states $\phi = \sum_{k\in K} k\omega k^*$ for some finite set of Kraus operators $K \subset \mathcal{A}^{loc}$. For such a state, an implementing density operator is given by

$$\rho_\phi = \sum_{k\in K} \pi(k)|\Omega\rangle\langle\Omega|\pi(k)^*. \tag{45}$$

(Exercise!) By Kaplansky's density theorem (Theorem 2.8), the contractions in $\pi(\mathcal{A}^{loc})$ are strongly dense in the unit ball of $B(\mathcal{H})$. This implies $\|\rho_\phi - \rho_\psi\|_1 = \|\phi - \psi\|$. Therefore, a Cauchy sequence (ϕ_n) gives a Cauchy sequence (ρ_{ϕ_n}). Since $\mathcal{S}_0$ is norm dense in $\mathcal{S}$ (see (38)), this implies that every state ϕ in $\mathcal{S}$ is implemented by a density operator ρ_ϕ. The results below (see Proposition 4.2) imply that every density operator on $\mathcal{H}$ corresponds to some state in the folium $\mathcal{S}$, so that we have a bijection

$$\mathcal{S} \ni \phi \longleftrightarrow \rho_\phi \in \mathcal{D}(\mathcal{H}). \tag{46}$$

From now on, we work in a fixed pure state sector $\mathcal{S}$. We consider the corresponding Hilbert space $\mathcal{H}$ and identify the local algebra $\mathcal{A}^{loc}$ with $\pi(\mathcal{A}^{loc})$, the quasi-local algebra $\mathcal{A}$ with $\pi(\mathcal{A})$, and the sector $\mathcal{S}$ with $\mathcal{D}(\mathcal{H})$, the set of density operators on $\mathcal{H}$.

4.D Von Neumann algebras of subsystems. To a region $A \subseteq \Gamma$, we associate the von Neumann algebra

$$\mathcal{M}_A = \overline{\pi(\mathcal{A}^{loc}_A)}^{sot} \subseteq B(\mathcal{H}). \tag{47}$$

Since strong convergence preserves commutativity, we have

$$A \cap B = \emptyset \qquad \Longrightarrow \qquad \mathcal{M}_A \subseteq \mathcal{M}'_B, \tag{48}$$

and since the *-algebra generated by a family $\mathcal{A}^{loc}_{A_i}$, $i \in I$, is $\mathcal{A}^{loc}_{\cup_i A_i}$, we have

$$\vee_i \mathcal{M}_{A_i} = \mathcal{M}_{\cup_i A_i}. \tag{49}$$

The irreducibility of the GNS representation gives $\mathcal{M}_\Gamma = B(\mathcal{H})$; see (43). As a consequence, for each region $A \subseteq \Gamma$, the resulting von Neumann algebra is actually a *factor*:

$$Z(\mathcal{M}_A) = \mathcal{M}_A \cap \mathcal{M}'_A \subseteq \mathcal{M}'_{A^c} \cap \mathcal{M}'_A = (\mathcal{M}_{A^c} \vee \mathcal{M}_A)' = (\mathcal{M}_\Gamma)' = B(\mathcal{H})' = \mathbb{C}. \tag{50}$$

4.E State transitions in terms of von Neumann algebras. Our goal is to show that approximate state transitions

$$\psi \xrightarrow{A} \phi$$

with finite operations can be expressed using the von Neumann algebra $\mathcal{M}_A$ alone. We begin with the simpler case of unitary transitions.

Proposition 4.1. *Consider unit vectors $\Psi, \Phi \in \mathcal{H}$ and let $A \subseteq \Gamma$ be some region. The following are equivalent:*

(a) For every $\varepsilon > 0$, there exists a finite region $A_\varepsilon \Subset A$ and a unitary $u_A \in \mathcal{A}_{A_\varepsilon}$ such that

$$\|u_A\Psi - \Phi\| < \varepsilon. \tag{51}$$

(b) For every $\varepsilon > 0$, there exists a unitary $u_A \in \mathcal{M}_A$ such that (51) holds.

Proof. (a) ⇒ (b) is clear. For the converse, let $u_A \in \mathcal{M}_A$ be a unitary. Since $\mathcal{A}_A^{loc}$ is a unital *-algebra that generates $\mathcal{M}_A$, Kaplansky's density theorem (see Theorem 2.8) lets us approximate $u_A\Psi$ by $a\Psi$ using contractions $a \in \mathcal{A}_A^{loc}$. By Lemma 2.7, we can in turn approximate $a\Psi$ using a unitary $\tilde{u}_A$ in $\mathcal{A}_A^{loc}$. Since all errors can be made arbitrarily small, we are done. □

We now turn to general quantum channels.

Proposition 4.2. *Consider density operators ρ, σ on $\mathcal{H}$ and a region $A \subset \Gamma$. The following are equivalent:*

(a) $\rho \xrightarrow{A} \sigma$, i.e., for every $\varepsilon > 0$, there exists a finite region $A_\varepsilon \Subset A$ and a quantum channel T_ε with Kraus operators in $\mathcal{A}_{A_\varepsilon}$ such that

$$\|T_\varepsilon(\rho) - \sigma\|_1 < \varepsilon. \tag{52}$$

(b) For every $\varepsilon > 0$, there exists a quantum channel T_ε with finitely many Kraus operators in $\mathcal{M}_A$ such that (52) holds.

In the lecture, we will skip the proof, which again relies on Kaplansky's density theorem in a way similar to the proof of the unitary case.

Proof. (a) ⇒ (b) holds trivially. For the converse, we use Kaplansky's density theorem (see Theorem 2.8) applied to a matrix amplification. Let $T = \sum_{k\in K} k(\,\cdot\,)k^*$ be a quantum channel with finitely many Kraus operators $K \subset \mathcal{M}_A$. Pick an enumeration $K = \{k_1, \dots, k_n\}$ and consider the $\mathcal{M}_A$-valued matrix

$$v = \begin{pmatrix} k_1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ k_n & 0 & \cdots & 0 \end{pmatrix} \in M_n(\mathcal{M}_A), \qquad v^*v = \operatorname{diag}(1, 0, \dots, 0). \tag{53}$$

In particular, $\|v\| = 1$. The matrix amplification $M_n(\mathcal{M}_A)$ acts on $\mathcal{H} \otimes \mathbb{C}^n$. It is the von Neumann algebra generated by the unital *-subalgebra $M_n(\mathcal{A}_A^{loc})$. By Kaplansky's density theorem, there is a net of contractions $a_m \in M_n(\mathcal{A}_A^{loc})$ converging strongly to v. For each m, there is a finite region $A_m \Subset A$ with $a_m \in M_n(\mathcal{A}_{A_m})$. Denote the first-column entries by $a_{m,1}, \dots, a_{m,n}$. For each m, $\sum_j a_{m,j}^* a_{m,j} \leq 1$. We set

$$a_{m,n+1} = \left(1 - \sum_{i=1}^{n} a_{m,i}^* a_{m,i}\right)^{1/2} \in \mathcal{A}_{A_m}. \tag{54}$$

Then $T_m = \sum_{i=1}^{n+1} a_{m,i}(\,\cdot\,)a_{m,i}^*$ is a quantum channel acting only on A_m. For each $1 \leq i \leq n$, $\lim_m a_{m,i} = k_i$ strongly. We also claim that $a_{m,n+1} \to 0$ strongly. Indeed,

$$\|a_{m,n+1}\Psi\|^2 = \|\Psi\|^2 - \sum_{i=1}^{n} \|a_{m,i}\Psi\|^2 \xrightarrow{m} 0, \qquad \Psi \in \mathcal{H}. \tag{55}$$

For finite-rank density operators $\rho = \sum_{j=1}^{r} \lambda_j |\Psi_j\rangle\langle\Psi_j|$ on $\mathcal{H}$, this immediately implies convergence in trace norm:

$$T_m(\rho) = \sum_{i=1}^{n+1} a_{m,i}\rho a_{m,i}^* = \sum_{i=1}^{n+1}\sum_{j=1}^{r} \lambda_j |a_{m,i}\Psi_j\rangle\langle a_{m,i}\Psi_j| \xrightarrow{m} \sum_{i=1}^{n}\sum_{j=1}^{r} \lambda_j |k_i\Psi_j\rangle\langle k_i\Psi_j| = T(\rho). \tag{56}$$

Since any density operator can be approximated in trace norm by finite-rank ones and all the channels are trace-norm contractions, the triangle inequality finishes the proof. □

Working in the operational setting of finite operations and approximate state transitions, these results prove that we may always pretend that an agent with access to a region A can implement every quantum channel that has Kraus operators in the von Neumann algebra $\mathcal{M}_A$. In particular, the entanglement theory across a bipartition $\Gamma = A \cup B$ is completely encoded in the triple

$$(\mathcal{M}_A, \mathcal{M}_B, \mathcal{H}).$$

Consequently, it makes sense to study the entanglement theory for such triples abstractly.

4.F Haag duality. For a region $A \subset \Gamma$, we can ask whether Haag duality holds in the pure state sector under consideration, i.e., whether

$$\mathcal{M}'_A = \mathcal{M}_B, \qquad B = A^c. \tag{57}$$

In Theorem 3.1, we have shown that Haag duality is equivalent to uniqueness of purifications up to unitaries *in the von Neumann algebra* of the complementary subsystem. However, that is not what we want. We want finite support unitaries, i.e., unitaries in $\mathcal{A}_B^{loc}$. This is solved by Proposition 4.1 and weak density of $\mathcal{A}_A^{loc} \subset \mathcal{M}_A$. Together with Theorem 3.1, they show that the following are equivalent:

(a) *Haag duality:* $\mathcal{M}'_A = \mathcal{M}_B$.

(b) If $\Psi, \Phi \in \mathcal{H}$ are unit vectors with $\langle \Psi, a\Psi \rangle = \langle \Phi, a\Phi \rangle$ for all $a \in \mathcal{A}_A^{loc}$, then, for every $\varepsilon > 0$, there exists a unitary $u_B \in \mathcal{A}_B^{loc}$, such that $\|\Psi - u_B\Phi\| < \varepsilon$.

4.G Stability under finite perturbations. For a non-finite region $A \subset \Gamma$, $\mathcal{M}_A$ is not of type I_n, $n < \infty$. If we add finitely many sites F to a region A, then the von Neumann algebra changes by tensoring a finite type I factor

$$\mathcal{M}_{A \cup F} \cong \mathcal{M}_A \otimes \mathcal{M}_F. \tag{58}$$

For non-type I_n factors, tensoring with a finite type I factor does not change the type classification. It also cannot change the validity of Haag duality (see [6] or [54] for proofs). The same is true if one removes finitely many sites from A. Combining both, we see that neither Haag duality nor the type classification change when infinite regions are changed in a finite way, see Fig. 5.

This has an important conceptual consequence for those entanglement properties that are encoded in the type classification and Haag duality: These entanglement properties are invariant under finite perturbations of what counts as Alice and what counts as Bob. Inspired by this, in [6], we called these the *large-scale entanglement* properties of the system.

5 Examples

This section discusses von Neumann algebraic properties of concrete systems with infinitely many degrees of freedom.

5.A Infinitely many entangled pure states. We consider a setting where two agents A and B share infinitely many entangled qudit pairs. We may think of this as a quantum spin system, where the set Γ consists of two spin chains with neighboring A- and B-spins being in an entangled pure state $\Psi_x \in \mathbb{C}^d \otimes \mathbb{C}^d$, see Fig. 6.

Regarding this as a many-body system, we form the local algebra $\mathcal{A}^{loc}$ and use the GNS representation to construct a Hilbert space $\mathcal{H}$ from the product state ψ determined by (Ψ_x), as discussed

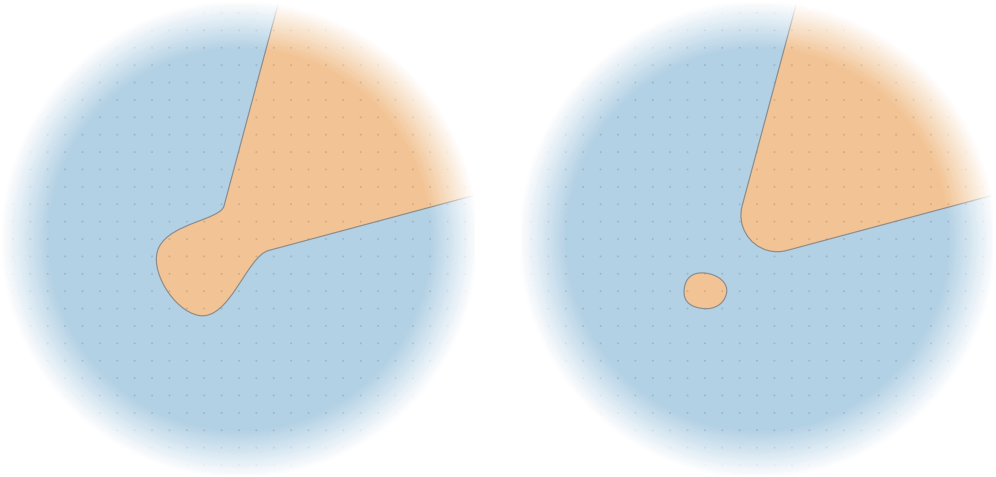


Figure 5: Two finite perturbations of a cone in a two-dimensional spin system. In both cases, the resulting von Neumann algebras have the same type and subtype, and Haag duality holds for one if and only if it holds for the other.

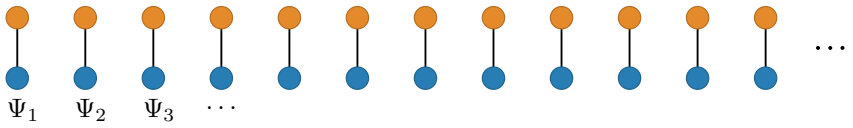


Figure 6: Infinitely many pairs of qudits shared between two agents. Each pair is in an entangled pure state Ψ_x, $x = 1, 2, \ldots$.

earlier. The resulting Hilbert space and von Neumann factors $\mathcal{M}_A$, $\mathcal{M}_B$ are well understood. The resulting Hilbert space is a special case of von Neumann's infinite tensor product of Hilbert spaces $\mathcal{H}_x = \mathbb{C}^d \otimes \mathbb{C}^d$, $x = 1, 2, \ldots$, relative to the sequence $(\Psi_x)_{x=1}^{\infty}$ [55], denoted

$$\mathcal{H} = \bigotimes_{x=1}^{\infty} (\mathbb{C}^d \otimes \mathbb{C}^d, \Psi_x). \tag{59}$$

It is constructed by noting that $\langle \otimes_x \Phi_x, \otimes_x \Phi'_x \rangle := \prod_x \langle \Phi_x, \Phi'_x \rangle$ extends to an inner product on linear combinations of vectors $\otimes_x \Phi_x$ where $\Phi_x = \Psi_x$ for all but finitely many x. Taking the completion then gives the Hilbert space (59).

The local algebras $\mathcal{A}_A^{loc}$ and $\mathcal{A}_B^{loc}$ are the unital *-algebras of operators acting only on finitely many A- or B-spins and the von Neumann algebras they generate are the factors $\mathcal{M}_A$ and $\mathcal{M}_B$. This construction is well known in the math literature. It is known that Haag duality between A and B holds unconditionally [57, Thm. XIV.1.9], that $\mathcal{M}_A$ and $\mathcal{M}_B$ are *-isomorphic (just swap A- and B-spins), and it is known how the type and subtype can be computed from the asymptotic scaling of the Schmidt spectra of the pure states Ψ_x [56]. The type is stable under perturbations that vanish asymptotically in a suitable sense. Using identical pure states $\Psi_x \equiv \Psi$ and $d \leq 3$, we obtain factors of types I_∞, II_1, II_∞, and III_λ for $0 < \lambda \leq 1$.[9] The relevant cases are summarized in the following table.

[9]The exotic type III_0 can be obtained from a suitable sequence (Ψ_n) of non-identical pure states, which become close to product states in the limit $n \to \infty$, but not too quickly, otherwise they give type I factors [56, Sec. 10].

	Schmidt spectrum	Type
Product states	$(1, 0, \dots, 0)$	I_∞
Maximally entangled	$(1/d, \dots, 1/d)$	II_1
Flat Schmidt spectrum	$(1/k, \dots, 1/k, 0, \dots, 0)$	II_∞
Generic qubit states	(p_1, p_2)	$\mathrm{III}_{\frac{p_2}{p_1}}$
Generic qutrit states	(p_1, p_2, p_3)	III_1

Table 1: Type and subtype of $\mathcal{M}_A$ and $\mathcal{M}_B$ for infinitely many identical pure states. For states with flat Schmidt spectrum, assume $1 < k < d$. The Schmidt spectrum is the spectrum of the reduced density operator, ordered non-increasingly. For generic qubit states, we assume $1 > p_1 > p_2 > 0$. For generic qutrit states, we assume $1 > p_1 > p_2 > p_3 > 0$ and $\frac{\log(p_1/p_3)}{\log(p_2/p_3)} \notin \mathbb{Q}$. All of these follow immediately from [39, 56].

5.B Translation-invariant spin chains. We consider a translation-invariant pure state ω on a spin chain. More specifically, ω is a pure state on the local algebra of a quantum spin system with $\Gamma = \mathbb{Z}$ and local Hilbert space dimension d, and we assume that ω is translation-invariant, i.e., it assigns the same expectation values to the translations of each finitely localized operator.

The entanglement entropy of a finite region $A \Subset \mathbb{Z}$ is $S_A = -\operatorname{tr}(\rho_A \log \rho_A)$, where ρ_A is the reduced density operator on $\otimes_{x \in A} \mathbb{C}^d$. We denote the entanglement entropy of an interval of length n by S_n (by translation-invariance, the entanglement entropy only depends on the length of the interval).

- *Area law implies type I.* In one spatial dimension, an area law means that the entanglement entropy is uniformly bounded $\sup_n S_n < \infty$. Matsui proved that this implies that the half-chain algebras $\mathcal{M}_L$ and $\mathcal{M}_R$ are type I factors [58]. In particular, pure gapped ground states of finite-range Hamiltonians have type I half-chain factors [58, 59].

- *Critical systems have type* III_1. In [16], it is shown that the critical XX model has type III_1 half-chain algebras in Haag duality. Later, in [5], we extended this to all models that can be mapped to free fermions with at least one energy band passing through zero. In general, we expect all critical models to have type III_1 factors, but this slogan is more folklore than a definite fact.

- *Vanishing entropy density and half-chain Haag duality.* In a recent preprint, Ruizhi Liu and I showed that half-chain Haag duality holds in the sector of a translation-invariant pure state if and only if the entropy density vanishes, $\lim_n S_n/n = 0$ [54].

5.C 2D spin systems. In two spatial dimensions (and higher), the interplay between algebraic properties and quantum matter properties is very rich and *a lot* can happen. This is an active area of current research [60–65]. This interplay gives us a wonderful playground for von Neumann algebraic quantum information theory: We can derive information-theoretic differences between distinct models, and it allows us to find physical realizations of abstract algebraic settings.

Type I algebras essentially only occur for product states, and unique gapped ground states of local finite-range translation-invariant Hamiltonians can exhibit all possible algebra types (except

possibly type III_0) already for the simplest geometric regions [6].

In general, Haag duality depends on the region and sector under consideration. Haag duality for cones is of particular importance to the mathematical physics community working on quantum matter because it underlies the operator-algebraic approach to constructing the categories of topological charges and understanding the fusion and braiding rules [48, 50, 66, 67]. For this reason, people worked hard to prove Haag duality of cone regions in exactly solvable models [60, 66, 68, 69].

In [54], for systems satisfying an area law with subleading corrections, we showed that Haag duality holds for all cones, but that it does not hold for unions of two or more disjoint cones unless the topological entanglement entropy [70–72] vanishes.

In [65], Wallick and Wilming showed that Haag duality fails for every cone in a sector with a single non-abelian anyon.

5.D Relativistic quantum fields. In algebraic quantum field theory, each spacetime region $\mathcal{O}$ is assigned a von Neumann algebra $\mathcal{A}(\mathcal{O}) \subseteq \mathcal{B}(\mathcal{H})$ of observables localized in $\mathcal{O}$, acting on the Hilbert space of a fixed sector. The map $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$ respects inclusions of regions, and algebras of causally separated regions commute. Under standard assumptions, the local algebras are AFD type III_1 factors and, hence, all isomorphic [73]. The properties of the quantum field theory at hand are encoded in the relative position of these algebras for different regions.

An operational interpretation of state transitions with operations that have Kraus operators in $\mathcal{A}(\mathcal{O})$, as we derived it for quantum spin systems, is problematic. The issue is that arbitrary operations or measurements described by operators in $\mathcal{A}(\mathcal{O})$ do not admit a physical implementation within $\mathcal{O}$. Indeed, assuming they do leads to causality issues in the form of super-luminal signaling [74, 75].

6 Pure state entanglement theory

When considering bipartitions of infinite quantum systems, we will often encounter sectors in which every pure state is infinitely entangled, i.e., has infinite one-shot distillable entanglement. Nonetheless, *comparative entanglement theory* ("this state is more entangled than that state"), defined in terms of state transitions that are possible via local operations and classical communication (LOCC), still makes sense.

We consider complementary subsystems A and B, described by commuting factors $\mathcal{M}_A$ and $\mathcal{M}_B$ on a Hilbert space $\mathcal{H}$. Throughout this section, we *assume Haag duality* holds: $\mathcal{M}_A = \mathcal{M}_B'$. An LOCC protocol is a protocol of the following form: Alice starts by performing an operation with (finitely many) Kraus operators $\{k_{i_1}\}$ in $\mathcal{M}_A$ and then communicating the outcome to Bob. Conditioned on Alice's outcome, Bob applies an operation with Kraus operators $\{k_{i_2|i_1}\} \subset \mathcal{M}_B$ and communicates his outcome to Alice, who performs an operation conditioned on Bob's outcome, etc. The protocol stops after finitely many rounds. For a string of outcomes $I = i_1, \dots i_n$, if the system was initially in a pure state Ψ, the final state Φ_I is given by

$$\Phi_I \propto k_I \Psi, \qquad k_I = k_{i_n|i_1,\dots,i_{n-1}} \cdots k_{i_2|i_1} k_{i_1}. \tag{60}$$

The average outcome state is the mixed state with density operator $\rho = \sum_I p_I |\Phi_I\rangle\langle\Phi_I|$, where $p_I = \|k_I \Psi\|^2$. We will mostly consider pure state entanglement theory here. We say that Ψ is more entangled than Φ or that Φ is reachable from Ψ using approximate LOCC, denoted

$$\Psi \xrightarrow{\text{LOCC}} \Phi, \tag{61}$$

if for every $\varepsilon > 0$, there exists an LOCC protocol whose output state ρ_ε (under Ψ as the input state) is ε-close to Φ, i.e.,

$$\| |\Phi\rangle\langle\Phi| - \rho_\varepsilon \|_1 < \varepsilon. \tag{62}$$

Of course, the definition generalizes to mixed states.

In finite dimensions, Nielsen's theorem characterizes pure-state LOCC transitions by *majorization* of the spectrum of the reduced density operators (the Schmidt decomposition guarantees that the two reduced density operators have the same spectrum) [76]. For normal states ψ, ϕ on a factor $\mathcal{M}$, we write $\psi \preceq \phi$ if ψ can be approximated using convex combinations of unitary rotations of ϕ:

$$\psi \in \overline{\text{conv}}^{\|\cdot\|}\{u\phi u^* : u \in U(\mathcal{M})\}. \tag{63}$$

In the type I case, this is equivalent to majorization of the spectra of the reduced states. In type II, spectra of reduced states can be defined and a similar result can be shown (see our paper [3] or my thesis [26] for a discussion in the context of entanglement theory). In type III, the story is both much more complicated and completely trivial. We will discuss its surprising consequence below.

Using the definition of majorization based on convex combination of unitaries, Nielsen's theorem extends to general factors:

Theorem 6.1 (Nielsen's theorem, [3, 77])**.** *Let $\Psi, \Phi \in \mathcal{H}$ be unit vectors, with marginals ψ_A, ϕ_A on $\mathcal{M}_A$. Then*

$$\Psi \xrightarrow{\text{LOCC}} \Phi \quad \iff \quad \psi_A \preceq \phi_A \quad \iff \quad \psi_B \preceq \phi_B. \tag{64}$$

For II factors, the theorem was previously obtained by Crann et al. [77]; the type III case was supplied in [3]. Haag duality is a crucial ingredient in the proof. Among other things, it is needed to show $\psi_A \preceq \phi_A \iff \psi_B \preceq \phi_B$. We do not know whether a failure of Haag duality implies that Nielsen's theorem is false.

6.A Pure-state LOCC trivializes in type III. In type III factors, every normal state can be approximated by convex combinations of unitary rotations of every other normal state [41, 78]. In other words, majorization $\psi \preceq \phi$ holds for any two normal states ψ, ϕ. Nielsen's theorem therefore gives [3]:

Corollary 6.2. *If $\mathcal{M}_A$ and $\mathcal{M}_B$ have type* III*, then*

$$\Psi \xrightarrow{\text{LOCC}} \Phi \qquad \textit{for all unit vectors } \Psi, \Phi \in \mathcal{H}. \tag{65}$$

If (65) holds, then either $\mathcal{M}_A$ and $\mathcal{M}_B$ have type III*, or $\mathcal{M}_A = \mathbb{C}$ and $\mathcal{M}_B = B(\mathcal{H})$, or $\mathcal{M}_A = B(\mathcal{H})$ and $\mathcal{M}_B = \mathbb{C}$.*

For type II factors, the LOCC ordering is nontrivial, although every pure state has infinite one-shot entanglement.

6.B Infinite one-shot entanglement. We ask how much entanglement A and B can extract from a single pure state $\Psi \in \mathcal{H}$. We care about the one-shot setting, not asymptotic rates.

We begin by describing what it means that a finite-dimensional bipartite state $\sigma_{A'B'}$ can be distilled. Alice and Bob attach finite-dimensional systems A' and B', respectively, initialized in the product state $|1\rangle_{A'} \otimes |1\rangle_{B'}$. Their von Neumann algebras then become

$$\mathcal{M}_{AA'} = \mathcal{M}_A \otimes B(\mathcal{H}_{A'}) \otimes 1_{B'}, \qquad \mathcal{M}_{BB'} = \mathcal{M}_B \otimes 1_{A'} \otimes B(\mathcal{H}_{B'}). \tag{66}$$

We say that a density operator $\sigma_{A'B'}$ can be *distilled* from Ψ in one shot if, for every $\varepsilon > 0$, there exists an LOCC protocol whose output state ρ_ε is ε-close to $\sigma_{A'B'}$:

$$\| \operatorname{tr}_{\mathcal{H}} \rho_\varepsilon - \sigma \|_1 < \varepsilon. \tag{67}$$

We say that Ψ has infinite one-shot entanglement if every finite-dimensional bipartite state can be distilled from it. Equivalently, if arbitrarily many Bell pairs can be distilled in one shot.

In a bipartite system of type I_∞ factors, i.e., when Alice and Bob each have an infinite-dimensional Hilbert space, each unit vector can only distill finitely many Bell pairs. However, there is no uniform upper bound on this number.

Using Nielsen's theorem and some results from majorization theory on type II and type III factors, the following connection to the type classification was shown in [3]:

Corollary 6.3. *A unit vector $\Psi \in \mathcal{H}$ has infinite one-shot entanglement if and only if $\mathcal{M}_A$ and $\mathcal{M}_B$ are of type* II *or* III*. In particular, if one unit vector in $\mathcal{H}$ has infinite one-shot entanglement, then every unit vector does.*

In the type III case, the proof is clear from the trivialization of LOCC. In the type II, the result is ultimately due to the non-existence of minimal projections. A sequence of smaller and smaller projections corresponds to a sequence of bipartite with less and less entanglement. This leftover entanglement is accessible using LOCC and can be used distill arbitrary finite-dimensional entangled state (see [3] or [26] for a proper proof.)

6.C Classification of types I_n, II_1, II_∞, and III. We call a unit vector $\Omega \in \mathcal{H}$ *maximally entangled* if

$$\Omega \xrightarrow{\text{LOCC}} \Psi \qquad \text{for every unit vector } \Psi \in \mathcal{H}. \tag{68}$$

For type I and II systems, maximally entangled states exist if and only if at least one of the local factors is finite [3, Prop. 58].

Recall that $\mathcal{M}_A$ is type I, II, or III if and only if $\mathcal{M}'_A = \mathcal{M}_B$ is. If the subtypes do not agree, we consider the *minimal type* of the pair. In type I, this is I_n with $n = \min\{n_A, n_B\}$, where $\mathcal{M}_A$ and $\mathcal{M}_B$ have types I_{n_A} and I_{n_B}. In type II, it is II_1 if at least one factor is finite, and II_∞ if both are infinite. In type III, the subtype of $\mathcal{M}_A$ equals that of $\mathcal{M}_B$.

The operational classification is then as follows [26, Thm. 7.18]: We define the distillable rank of a pure state $\Psi \in \mathcal{H}$ as the largest number d such that the maximally entangled 2-qudit state $\Phi_d \in \mathbb{C}^d \otimes \mathbb{C}^d$ can be distilled from it in one shot. If the state has infinite one-shot entanglement, the distillable rank is $d = \infty$.

Minimal type	Distillable rank	$\exists$ max. entangled state?	LOCC trivial
I_n, $n < \infty$	$\leq n$	yes	no (if $n > 1$)
I_∞	finite, but unbounded	no	no
II_1	infinite	yes	no
II_∞	infinite	no	no
III	infinite	yes (every state)	yes

For minimal type I_n, $n < \infty$, the bound $d \leq n$ is attained by a maximally entangled state. For minimal type I_∞, the distillable rank is finite, but unbounded.

7 Embezzlement of entanglement and the classification of type III factors

Embezzlement of entanglement is a phenomenon discovered by van Dam and Hayden in [79]. Roughly speaking, they found entangled "resource" states that allow one to create entangled states using local operations and no communication, without changing the resource state up to arbitrarily small errors.

They found a sequence of pure states Ω_n in finite-dimensional bipartite systems A_nB_n (the local dimension increases in n) such that, for *every* finite-dimensional bipartite pure state $\Psi_{A'B'}$ and every $\varepsilon > 0$, and all sufficiently large n, there exist unitaries $u_{A_nA'}$, $u_{B_nB'}$ such that

$$(u_{A_nA'} \otimes u_{B_nB'})(\Omega_n \otimes |1\rangle_{A'}|1\rangle_{B'}) \approx_\varepsilon \Omega_n \otimes \Psi_{A'B'}. \tag{69}$$

Here, smaller errors and more highly entangled states require larger n. It follows that, for any two pure states $\Phi_{A'B'}, \Psi_{A'B'}$, all $\varepsilon > 0$, and sufficiently large n, one can find unitaries such that

$$(u_{A_nA'} \otimes u_{B_nB'})(\Omega_n \otimes \Phi_{A'B'}) \approx_\varepsilon \Omega_n \otimes \Psi_{A'B'}. \tag{70}$$

It is clear that the error can never be made arbitrarily small for a fixed finite-dimensional pure state Ω, but it is suggestive to expect that the sequence of finite-dimensional states could be replaced by a bipartite system with infinitely many degrees of freedom with a resource state Ω for which the error ε can be made arbitrarily small. However, no such state can exist in the form of a bipartite pure state on a tensor product of infinite-dimensional Hilbert spaces. In fact, not even bipartite systems of type II factors can host such a state, but some type III factors can [1]. We will see that the classification of type III factors can be formulated entirely in terms of how well one can embezzle entanglement.

7.A Embezzling states. As in the previous section, we consider a bipartite system of commuting factors $\mathcal{M}_A$ and $\mathcal{M}_B$ on a Hilbert space $\mathcal{H}$, and we assume Haag duality $\mathcal{M}_A = \mathcal{M}_B'$.

We call a unit vector $\Omega \in \mathcal{H}$ *embezzling* if, for every $d \in \mathbb{N}$, every pair of unit vectors $\Phi, \Psi \in \mathbb{C}^d \otimes \mathbb{C}^d$, and every $\varepsilon > 0$, there exist local unitaries such that

$$\|u_{AA'}u_{BB'}(\Omega \otimes \Phi) - \Omega \otimes \Psi\| < \varepsilon. \tag{71}$$

Here $u_{AA'}$ is an element of $\mathcal{M}_A \otimes B(\mathbb{C}^d) \otimes 1$ and $u_{BB'}$ of $\mathcal{M}_B \otimes 1 \otimes B(\mathbb{C}^d)$. In particular, Φ may be a product state and Ψ arbitrarily entangled.

A strict separation into embezzling states and non-embezzling states is insufficient to recover the classification of type III factors. Instead, we need to *quantify* how well one can embezzle entanglement from a given state Ω. The ideal way to do this would be to consider the optimal fidelity up to which local unitaries can implement arbitrary state transitions $\Omega \otimes \Phi \to \Omega \otimes \Psi$. For technical reasons, we took another route in [1], where the quantification is based on the norm distance of the *marginal states*. This allows us to reduce the problem to a problem involving the norm distance of the reduced states on one of the two systems.[10]

Clearly the performance of a pure state Ω at embezzlement only depends on the local unitary orbit. By uniqueness of purifications, this means that the embezzlement performance is completely determined by the unitary orbit of the marginal state ω_A alone.

Given Ω, Ψ, Φ as above, we denote their respective marginal states by ω_A, $\psi_{A'}$, $\phi_{A'}$. We set

$$\kappa_A(\Omega; \Phi, \Psi) := \inf_{u_{AA'}} \|u_{AA'}(\omega_A \otimes \phi_{A'})u_{AA'}^* - \omega_A \otimes \psi_{A'}\|. \tag{72}$$

[10] The analogous reduction can also be done with the fidelity, but the resulting task is harder to solve [26].

This measures how well Alice can distinguish the output from the goal state after optimizing her unitary.

The quantifier $\kappa_A(\Omega;\Phi,\Psi)$ cannot be changed using unitaries $u_{BB'}$. If it is small then Uhlmann's theorem [26, 80–82] supplies a unitary $u_{BB'}$ so that $u_{AA'}u_{BB'}(\Omega\otimes\Phi)\approx\Omega\otimes\Psi$. (The error trade-off is governed by the Fuchs-van de Graaf inequalities [83].) In particular, it is zero if and only if the state transition $\Phi\to\Psi$ can be "embezzled" from Ω:

$$\kappa_A(\Omega;\Phi,\Psi)=0 \quad\iff\quad \forall\varepsilon>0\ \exists u_{AA'},u_{BB'}: u_{AA'}u_{BB'}(\Omega\otimes\Phi)\approx_\varepsilon \Omega\otimes\Psi. \tag{73}$$

Although defined in an A-B asymmetric way, the error above is symmetric [26, Secs. 6.3, 8.1]:

$$\kappa_A(\Omega;\Phi,\Psi)=\kappa_B(\Omega;\Phi,\Psi). \tag{74}$$

We define the *uniform* error by

$$\kappa(\Omega):=\sup_d\sup_{\Phi,\Psi}\kappa_A(\Omega;\Phi,\Psi)=\sup_d\sup_{\Phi,\Psi}\kappa_B(\Omega;\Phi,\Psi), \tag{75}$$

where the inner suprema run over pairs of unit vectors in $\mathbb{C}^d\otimes\mathbb{C}^d$. The uniform error is small if and only if every transition $\Phi\to\Psi$ can be embezzled using Ω up to a small error. In particular, the state Ω is embezzling if and only if $\kappa(\Omega)=0$.

7.B Classification of type III_λ, $0\le\lambda\le1$. To obtain a property of the whole bipartite system, we take the worst performance among all resource states:

$$\kappa_{\max}(\mathcal{M}_A,\mathcal{M}_B,\mathcal{H}):=\sup_\Omega\kappa(\Omega), \tag{76}$$

where the sup is over unit vectors in $\mathcal{H}$. For type III factors, the main result of [1], based on a beautiful toolkit developed by Haagerup and Størmer in [41], identifies this quantity with the diameter of the state space modulo unitaries (see (20)):

Theorem 7.1 ([1])**.** *In a bipartite system of type* III *factors in Haag duality,*

$$\kappa_{\max}(\mathcal{M}_A,\mathcal{M}_B,\mathcal{H})=\operatorname{diam}\mathcal{S}_*(\mathcal{M}_A)/_\sim=\operatorname{diam}\mathcal{S}_*(\mathcal{M}_B)/_\sim\,. \tag{77}$$

For type III factors, knowing the diameter is equivalent to knowing the subtype. In fact, in these lecture notes we have taken the state space diameter as the definition of the classification of type III factors. By Eqs. (21) and (77), the subtype of a bipartite system of type III factors and $\kappa_{\max}=\kappa_{\max}(\mathcal{M}_A,\mathcal{M}_B,\mathcal{H})$ are related via

$$\lambda=\left(\frac{2-\kappa_{\max}}{2+\kappa_{\max}}\right)^2,\qquad \kappa_{\max}=2\,\frac{1-\sqrt{\lambda}}{1+\sqrt{\lambda}}. \tag{78}$$

We say that a bipartite system is a universal embezzler if every pure state is embezzling. We then have:

Corollary 7.2 ([1])**.** *A bipartite system of factors satisfying Haag duality is a universal embezzler if and only if its local factors have type* III_1*.*

For type I and II systems, by contrast, $\kappa(\Omega)=2$ for every unit vector Ω. Thus, infinite one-shot entanglement, which is present in both types II and III, does not by itself imply embezzlement.

For more details about embezzlement of entanglement in the setting of von Neumann algebras, see our paper [1] or my thesis [26].[11]

[11] Embezzlement of entanglement was also studied using operator algebras in a series of great papers that should be mentioned here, although they are mostly unrelated to the operational setting that we considered here [19, 84, 85].

8 Outlook

We began these lectures by asking whether there are operationally distinct forms of infinite entanglement. The connections between state transitions, Haag duality, and the type classification give several answers to this question. These connections allow us to translate algebraic properties of concrete physical models into entanglement properties, and conversely to interpret operational properties algebraically. There is much more to explore in both directions.

In bipartite systems of factors in Haag duality, many aspects of finite-dimensional quantum information theory generalize. We mentioned a few here, but skipped many others. A more complete detailed, but far from complete, account is given my PhD thesis [26]- Examples include Uhlmann's theorem, Radon-Nikodym results, stochastic LOCC transitions and the Schmidt rank, etc.

An important limitation of our discussion in Sections 6 and 7 is the assumption of Haag duality in the entanglement results. When Haag duality fails, the types and subtypes of $\mathcal{M}_A$ and $\mathcal{M}_B$ do not determine the relative position of the two algebras, not even when tomographic completeness, i.e., $\mathcal{M}_A \vee \mathcal{M}_B = B(\mathcal{H})$, still holds. In this case, the algebraic object to study is the subfactor inclusion

$$\mathcal{M}_A \subseteq \mathcal{M}_B'. \tag{79}$$

We already know that some useful operational statements remain possible even when Haag duality fails. We recently studied the Hughston-Jozsa-Wootters theorem [86] for bipartite systems of commuting factors $\mathcal{M}_A, \mathcal{M}_B$ on a Hilbert space. In finite-dimensional systems, the theorem asserts that Bob can prepare any ensemble decomposition of Alice's marginal of a global pure state by measuring his subsystem, We showed that whether it holds in the von Neumann-algebraic setting depends on the relative position of the algebras $\mathcal{M}_A$ and $\mathcal{M}_B$ (even if they satisfy tomographic completeness $\mathcal{M}_A \vee \mathcal{M}_B = B(\mathcal{H})$) and on the choice of the global pure state. More specifically, it holds if and only if the subfactor (79) admits a so-called conditional expectation, which preserves the marginal states [18]. Trivially, this is the case when Haag duality holds, but there are several interesting examples where it holds despite Haag duality being false, and there are other interesting examples where it fails. Most other questions about the entanglement theory of bipartite systems without Haag duality remains open.

In a multipartite system, the types and subtypes of the individual subsystem algebras do not encode the genuinely multipartite entanglement structure. This requires information about the relative positions of several algebras acting on the same Hilbert space. Meaningful and nontrivial statements on multipartite entanglement in system of several commuting factors are very hard to obtain. This is not surprising given how complex genuinely multipartite entanglement is already in finite-dimensional systems. We have some results on *multipartite embezzlement*: universal multipartite embezzlers exist as abstract systems for any finite number of parties, but we do not know natural physical realizations of these abstract systems [4].

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