October 15, 2026

Exercise sheet:
Entanglement in infinite quantum spin systems

These exercises concern the structure of states, state transitions, and entanglement in infinite quantum spin systems. As in the lecture, Γ is a countably infinite set, and at each site $x \in \Gamma$, we imagine a d-dimensional quantum system. The local algebra of finitely supported operators is denoted $\mathcal{A}^{loc}$.

1. Purity of algebraic states. A state of the quantum spin system is a linear functional $\omega : \mathcal{A}^{loc} \to \mathbb{C}$ that is positive, i.e., $\omega(a^*a) \geq 0$ for $a \in \mathcal{A}^{loc}$, and unital $\omega(1) = 1$.

(a) Show that the state space $\mathcal{S}(\mathcal{A}^{loc})$ is a convex set: If $p \in [0,1]$ and ω_1, ω_2 are states, then so is $p\omega_1 + (1-p)\omega_2$.

A state ω is called *pure* if it is an extremal point of the state space, i.e., if $\omega = p\omega_1 + (1-p)\omega_2$ for $p \in (0,1)$ and $\omega_{1,2} \in \mathcal{S}(\mathcal{A}^{loc})$ implies $\omega_1 = \omega_2 = \omega$.

(b) Show that a state ω is pure if and only if every positive linear functional $\phi : \mathcal{A}^{loc} \to \mathbb{C}$ with $\phi \leq \omega$ (i.e., $\omega - \phi$ is positive) is of the form $\phi = t\omega$ for some $t \in [0,1]$.

(c) Let $A \subset \Gamma$ be a subregion. The reduced state on the region A is $\omega_A := \omega|_{\mathcal{A}_A^{loc}} \in \mathcal{S}(\mathcal{A}_A^{loc})$. Show that if ω_A is a pure state, then ω is a product state:

$$\omega(ab) = \omega(a)\omega(b), \qquad a \in \mathcal{A}_A^{loc},\ b \in \mathcal{A}_{\Gamma\setminus A}^{loc}, \tag{1}$$

(Hints: every element of $\mathcal{A}_X$ is a linear combination of positive elements; use item (b).)

2. What changes when errors need only vanish pointwise? The operational setting we discussed allows an agent with access to a region $A \subset \Gamma$ of a spin system only to act on finite regions. For a state transition from ϕ to ψ, we have demanded that, by acting on finite regions, the agent can make errors in expectation values vanish uniformly: For all $\varepsilon > 0$, there exists a finite region $A_0 \subset A$ and operators $k_1, \dots k_n \in \mathcal{A}_{A_0}$ with $\sum_i k_i^* k_i = 1$ such that

$$\left|\psi(a) - \sum_{i=1}^{n} \phi(k_i^* a k_i)\right| < \varepsilon, \qquad \textit{uniformly} \text{ in } a, \tag{2}$$

where a ranges over operators in $\mathcal{A}^{loc}$ with $\|a\| \leq 1$. In this exercise, we explore what happens if we weaken the assumptions further by dropping uniformity.

(a) Formulate the above criterion using norm-convergence of states on the local algebra $\mathcal{A}^{loc}$.

(b) One says that a sequence of states ψ_n converges to a state ψ in the weak*-topology if $\psi_n(a) \to \psi(a)$ for all $a \in \mathcal{A}^{loc}$. Formulate a non-uniform version of (2) using weak*-convergence.

(c) Write the lattice Γ as a union of increasing finite regions $\Gamma = \cup_n \Gamma_n$. Starting from the all-$|0\rangle$ state, flip the spins in the region Γ_n. Show that the resulting states weak*-converge to the all-$|1\rangle$ state. Does it also converge in norm?

(d) Investigate whether an agent with access to the full system can reach an arbitrary state ψ from an arbitrary initial state ϕ in the non-uniform version of finite operations and approximate state transitions. (Hint: What can finite channels do to the state on a finite region?)

3. Where did the entanglement go? Alice and Bob share infinitely many qubit pairs A_iB_i, $i \in \mathbb{N}$. You can think of this as the quantum spin system with $\Gamma = A \cup B$ and $A = \{(i,0) : i \in \mathbb{N}\}$, $B = \{(i,1) : i \in \mathbb{N}\}$, identifying $A_i = (i,0)$, $B_i = (i,1)$. Let ψ be the global pure state where A_1B_1 are maximally entangled $|\Phi\rangle_{A_1B_1} = \frac{1}{\sqrt{2}}(|0\rangle_{A_1}|0\rangle_{B_1} + |1\rangle_{A_1}|1\rangle_{B_1})$ and all other qubit pairs are in the product state $|0\rangle_{A_i}|0\rangle_{A_i}$. Alice applies finite-support unitaries u_i that swap A_1 and A_i (note that this unitary acts on a finite region). The resulting state is $u_i\psi u_i^* := \psi(u_i^* \cdot u_i)$.

(a) Does $u_i\psi u_i^*$ have a weak*-limit? Determine the limit.

(b) Each $u_i\psi u_i^*$ is pure and entangled, what about the limit?

(c) Show that the sequence $u_i\psi u_i^*$ does not converge in norm.