

Petz recovery and sufficiency
for positive maps

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work with Henrik Wilming

Motivation

Asymmetric Hypothesis testing: distinguish f from σ .

When can we say:

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It seems, the DPI holds for PTP maps. Why?

Could we define (1) via those?

The Recovery Problem

Dichotomy: states (ρ, σ) on a f -dim space $\mathcal{H}$
Standing assumption: $\sigma > 0$

Quantum relative entropy: $D(\rho \parallel \sigma) = \text{tr } \rho (\log \rho - \log \sigma)$

DPI: T positive trace-preserving (PTP) map
 $D(T(\rho) \parallel T(\sigma)) \leq D(\rho \parallel \sigma)$.

Problem. Does DPI-equality imply a recovery map R ?
 $R \circ T(\rho) = \rho$, $R \circ T(\sigma) = \sigma$

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Problem. Does **DPI-equality** imply a **recovery map** $\mathcal{R}$?
 $\mathcal{R} \circ T(\rho) = \rho$, $\mathcal{R} \circ T(\sigma) = \sigma$

Theorem (Petz). (1) Yes, if T is CPTP.

(2) If so, $\mathcal{R} = \sigma^{\frac{1}{2}} T^* (T(\sigma)^{-\frac{1}{2}} \cdot T(\sigma)^{-\frac{1}{2}}) \sigma^{\frac{1}{2}}$.

Positive maps force Jordan structure

Schrödinger picture

(C)PTP map T



Heisenberg picture

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Fixed points of UCP maps¹

$*$ -algebra $A \subset L(\mathcal{H})$

¹ with full-rank invariant state

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Fixed points of UP maps¹

J^* -algebra $\mathcal{J} \subset L(\mathcal{H})$

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Minimal sufficient observables

A self-adj unital subspace $M \subset L(\mathcal{H})$ is $\mathcal{P} / \mathcal{CP}$ sufficient
if $\exists$ PTP / CPTP map T s.t.

$$\text{range}(T^*) \subseteq M, \quad T(\beta) = \beta, \quad T(\sigma) = \sigma.$$

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In both cases, there exist minimal objects:

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$$J(\rho, \sigma)$$

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$$J(\rho, \sigma) \subseteq A(\rho, \sigma).$$

Bayesian sufficiency

Prior p for g and $1-p$ for σ .

A **test** is an observable $0 \leq x \leq \mathbb{1}$. Success probability:

$$P_{\text{success}, p} = p \operatorname{tr} x g + (1-p) \operatorname{tr} (\mathbb{1} - x) \sigma$$

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$\hookrightarrow \exists$ minimal Bayes-sufficient unital self-adj. subspace

$$M(\rho, \sigma) = \operatorname{span} \{ \mathbb{1}, [\rho > t \sigma] ; t > 0 \}$$

Structure Theorem

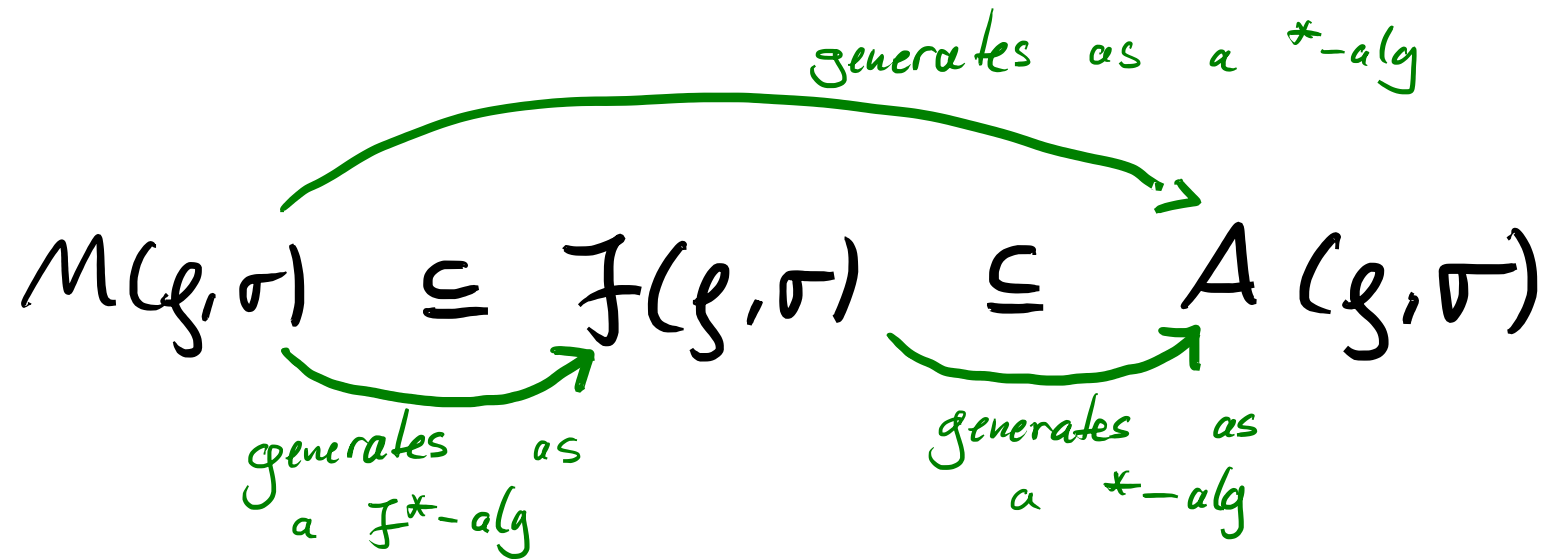
(1) Hierarchy

$$M(g, \sigma) \subseteq F(g, \sigma) \subseteq A(g, \sigma)$$

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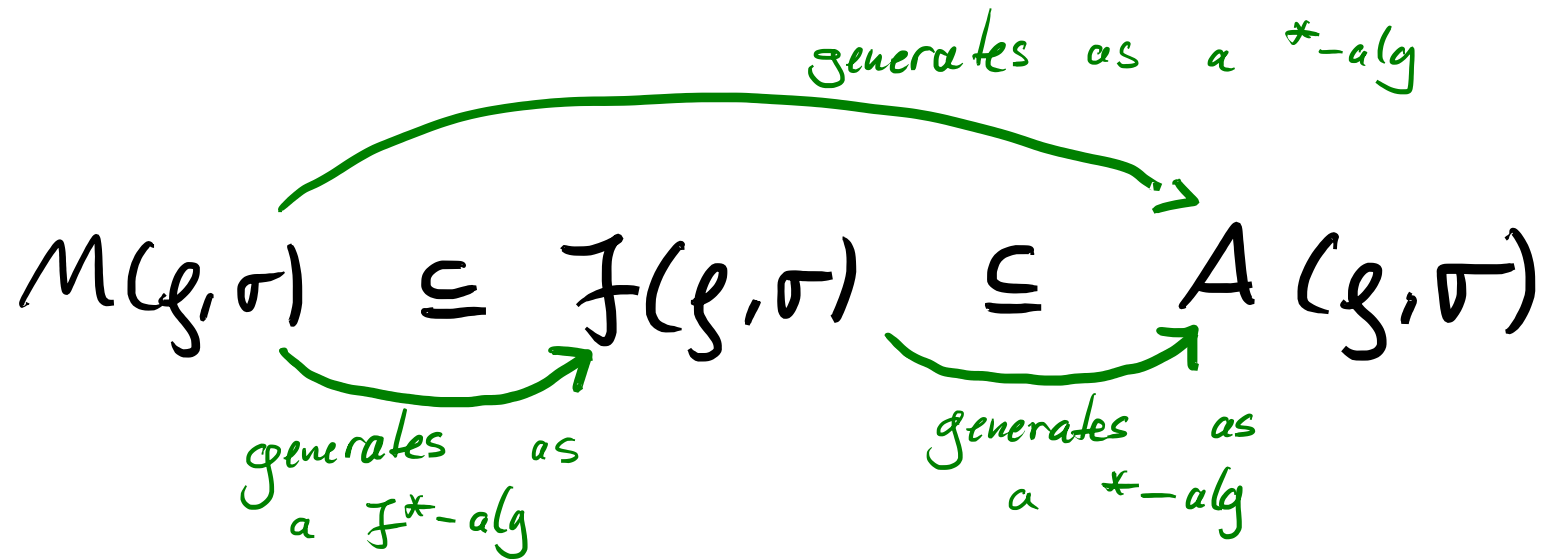
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→ Bayes sufficiency determines P & CP recovery theory!

From relative entropy to Bayesian tests

Frenkel's integral formula :

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$$\Leftrightarrow T^* \text{ maps } [T(\rho) \geq \epsilon T(\sigma)] \text{ to } [\rho \geq \epsilon \sigma] \quad \forall \epsilon > 0.$$

From DPl-equality to a $\mathcal{F}^*$ -isomorphism

Claim $T^*: \mathcal{F}(T(\varrho), T(\sigma)) \rightarrow \mathcal{F}(\varrho, \sigma)$ is a $\mathcal{F}^*$ -iso.

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$$\mathcal{L}(\mathcal{H}) \xrightarrow{R^*} \mathcal{L}(\mathcal{K})$$

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Recovery Theorem

Let $T: L(\mathcal{H}) \rightarrow L(\mathcal{K})$ be a PTP map. Then

DPI equality

$$\begin{aligned} D(T(\rho) \| T(\sigma)) \\ = D(\rho \| \sigma) \end{aligned}$$

$\Leftrightarrow$

Recovery

$\exists$ PTP map

$R: L(\mathcal{K}) \rightarrow L(\mathcal{H})$ s.t.

$$R \circ T(\rho) = \rho, \quad R \circ T(\sigma) = \sigma.$$

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Conclution / Outlook

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— Neither does **Recovery**!
- Same for the **sandwiched** and α - $\mathbb{Z}$ quantum Rényi divergence.
- If $(\rho, \sigma) \xleftrightarrow{\text{PTP}} (\tau, \omega)$, then one can interconvert using **decomposable** PTP maps. (CP + coCP).
↳ The transpose + CPTP maps determine PTP-equivalence.