Petz recovery and sufficiency for positive maps

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work with Henrik Wilming

Motivation

Asymmetric Hypothesis testing: distinguish ρ from σ.

When can we say:

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It seems, the DPI holds for PTP maps. Why?
Could we define (1) via those?

The Recovery Problem

Dichotomy: states (ρ, σ) on a f-dim space $\mathcal{H}$

Standing assumption: $\sigma > 0$

Quantum relative entropy: $D(\rho \| \sigma) = \operatorname{tr} \rho (\log \rho - \log \sigma)$

DPI: T positive trace-preserving (PTP) map

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<u>Problem.</u> Does DPI-equality imply a recovery map R?

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> Theorem (Petz). (1) Yes, if T is CPTP.
>
> (2) If so, $R = \sigma^{\frac{1}{2}} T^{*}\left(T(\sigma)^{-\frac{1}{2}} \cdot T(\sigma)^{-\frac{1}{2}} \right) \sigma^{\frac{1}{2}}.$

Positive maps force Jordan structure

Schrödinger picture $\longleftrightarrow$ Heisenberg picture

(C)PTP map T $\longleftrightarrow$ U(C)P map T^*

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$$T^*(a)T^*(a^\dagger) \leq T^*(a^\dagger a)$$

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Fixed points of UCP maps [1]

$*$-algebra $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$

[1] with full-rank invariant state

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Fixed points of UP maps[1]

J^*-algebra $J \subset L(\mathcal{H})$

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Minimal sufficient observables

A self-adj unital subspace $\mathcal{M} \subset \mathcal{L}(\mathcal{H})$ is **P / CP sufficient** if $\exists$ PTP / CPTP map T s.t.

$$\mathrm{range}(T^*) \subseteq \mathcal{M}, \qquad T(\rho) = \rho, \qquad T(\sigma) = \sigma.$$

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In both cases, there exist minimal objects:

$\mathcal{A}(\varrho, \sigma)$

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$\mathcal{J}(\rho, \sigma)$

minimal P-sufficient, $\mathcal{J}^*$-algebra

$$\mathcal{J}(\rho, \sigma) \subseteq \mathcal{A}(\rho, \sigma).$$

Bayesian sufficiency

Prior p for ϱ and $1-p$ for σ.

A **test** is an observable $0 \leq x \leq \mathbb{1}$. Success probability:

$$P_{\text{success},p} = p \operatorname{tr} x \varrho + (1-p) \operatorname{tr}(\mathbb{1} - x)\sigma$$

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$\hookrightarrow$ $\exists$ minimal Bayes-sufficient unital self-adj. subspace

$$\mathcal{M}(\rho,\sigma) = \operatorname{span}\{\mathbb{1}, [\rho > t\sigma] \,;\, t > 0\}$$

Structure Theorem

(1) Hierarchy

$$\mathcal{M}(\mathfrak{g},\sigma) \subseteq \mathcal{F}(\mathfrak{g},\sigma) \subseteq \mathcal{A}(\mathfrak{g},\sigma)$$

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(2) The optimal tests generate everything

$$\begin{tikzcd}
M(\varrho,\sigma) \arrow[r, phantom, "\subseteq"] \arrow[r, bend right, "\text{generates as a } \mathcal{J}^*\text{-alg}"'] \arrow[rr, bend left, "\text{generates as a } *\text{-alg}"] & \mathcal{J}(\varrho,\sigma) \arrow[r, phantom, "\subseteq"] \arrow[r, bend right, "\text{generates as a } *\text{-alg}"'] & \mathcal{A}(\varrho,\sigma)
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$\rightarrow$ Bayes sufficiency determines P & CP recovery theory!

From relative entropy to Bayesian tests

Frenkel's integral formula:

some positive measure

$$D(\rho\|\sigma) = \int_0^1 P^*_{succ,p}(\rho\|\sigma)\, d\mu(p)$$

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$$\Leftrightarrow \quad T^* \text{ maps } [T(\rho) > t\,T(\sigma)] \text{ to } [\rho > t\,\sigma] \qquad \forall t>0.$$

From DPI-equality to a $\mathcal{F}^*$-isomorphism

<u>Claim</u> $T^* : \mathcal{F}(T(\rho), T(\sigma)) \to \mathcal{F}(\rho, \sigma)$ is a $\mathcal{F}^*$-iso.

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<u>Claim</u> $T^* : \mathcal{F}(\,T(\rho), T(\sigma)\,) \to \mathcal{F}(\rho, \sigma)$ is a $\mathcal{F}^*$-iso.

Why?
- $\mathcal{F}(\rho,\sigma)$ is generated by $[\rho > t\,\sigma]$, $t>0$
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From DPI-equality to a $\mathcal{J}^*$-isomorphism

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Building the recovery map R :

$$\mathcal{L}(\mathcal{H}) \xrightarrow{\quad R^* \quad} \mathcal{L}(\mathcal{K})$$

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\end{tikzcd}$$

Recovery Theorem

Let $T : \mathcal{L}(\mathcal{H}) \longrightarrow \mathcal{L}(\mathcal{K})$ be a PTP map. Then

<u>DPI equality</u>

$$D(T(\rho) \| T(\sigma)) = D(\rho \| \sigma)$$

$$\Longleftrightarrow$$

<u>Recovery</u>

$\exists$ PTP map $\mathcal{R} : \mathcal{L}(\mathcal{K}) \longrightarrow \mathcal{L}(\mathcal{H})$ s.t.

$$\mathcal{R} \circ T(\rho) = \rho, \quad \mathcal{R} \circ T(\sigma) = \sigma.$$

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- If $(\rho, \sigma) \xleftrightarrow{PTP} (\tau, \omega)$, then one can interconvert using decomposable PTP maps. (CP + coCP).

  ↳ The transpose + CPTP maps determine PTP-equivalence.